J. Fluid Mech. (2009), *vol.* 623, *pp.* 209–240.
doi:10.1017/S0022112008005296 Printed in the United Kingdom

On the normal incidence of linear waves over a plane incline partially covered by a rigid lid

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(Received 30 April 2008 and in revised form 7 November 2008)

The effect is examined on infinitesimal standing waves over a plane beach when restricted by the arbitrary placing of a finite rigid (or permeable) lid of length ℓ on the undisturbed surface. A uniformly bounded solution for the potential function is obtained by a Green's function method. The Green's function is derived and manipulated, for subsequent computational expedience, from a previously known solution for the problem of an oscillating line source placed at an arbitrary location in the sector. Applications are made to both the case of plate anchored at the origin and the case of plate anchored some distance at sea (drifted plate problem). In both cases water column potentials and equipotentials are constructed from the numerical solution of a Fredholm equation of the second kind by finite difference discretization. Solutions are further extended to include the logarithmically singular standing wave, combination with which allows the construction of progressing waves. Computation of initially incoming progressing wave envelopes demonstrates the emergence of a partially standing wave pattern shoreward of the plate. There is no difficulty, in principle, to extend the theory to any number of plates, and this is verified by computation for the case of two plates. A new shoreline radiation condition is constructed to allow formulation, in the usual way, of the reflection/transmission problem for the plate, and results are in good qualitative agreement with a similar model on a horizontal plane bed. It is argued that the Green's function constructed here could be used in a number of diverse problems, of this linear nature, where all, or part, of the submerged boundary is that of a plane incline.

1. Introduction

The theory of infinitesimal plane waves normally (or obliquely) incident on a flat beach is well understood and documented within the framework of a small-amplitude linear perfect fluid (but non-hydrostatic) theory (see e.g. Stoker 1957; Roseau 1958; Ehrenmark 1987). This amounts essentially to the development of two independent standing waves which are out of phase by $\pi/2$ at infinity but of which only one is uniformly bounded, the other possessing a log singularity in potential at the shore. In the literature this singularity has however tended to stigmatize somewhat the solutions representing progressing waves, as these require a combination of the two, so much so that some researchers (e.g. Minzoni & Whitham 1977; Blondeaux & Vittori 1995) prefer to work solely with the bounded standing wave perhaps because they feel uncomfortable with the otherwise unrepresentative values very close to the shore. This restriction disables description of the motion in the form of progressing waves, which are those commonly observed beyond the break zone on most beaches. It is

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hoped that the present work, by its approach particularly to the reflection problem, will help expose the advantage of including the singular solution and so perhaps remove some of this stigma. In this respect, the reader may benefit from adopting the purely abstract notion that a beach problem on $[0, \infty]$ is transformed as in (Roseau 1976, pp. 312–328) on to an infinite strip, where $+\infty$ remains invariant but where the origin is mapped to $-\infty$. In this way one can perhaps more readily accept the notion that the 'amplitude' of the log singularity is somehow representative of a transmitted wave in a diffracting obstacle problem and that the concern about large values infinitesimally close to the origin can be cast aside. The disproportionate values are of course a consequence of the energy of that wave eventually being concentrated over gradually decreasing depth. The energy flux remains however a measurable quantity, and if e.g. viscosity or surface tension effects are included (see e.g. Ehrenmark 1992; Miles 1990), then the wave height could also remain measurable.

The 'dock problem' generalization to the beach problem, here envisaged, is achieved by constructing a Green's function particularly suited to the wedge problem with a mixed (Robin) condition on the surface and a Neumann (or Dirichlet) condition on the bed. Similar problems albeit for uniform or infinite depth and also for elastic or moveable sheets have been considered by others, e.g. Friedrichs & Lewy (1948), Heins (1948), Linton (2001), Linton & Chung (2003), Hermans (2003) and Chung & Linton (2005), although none of these fully computes the diffracted wave profiles (some amplitudes are displayed in Chung & Linton 2005). Understandably, at the time, the first two authors concentrated on developing theoretical expressions for respectively the semi-infinite and finite dock problems, whilst the others computed reflection and transmission coefficients in various situations. For the fixed non-permeable plate these coefficients are evidently monotonic, considered as functions of wavenumber $\times$ plate length. For elastic plates though the computations expose Bragg resonance in a manner not dissimilar to that due to the presence of bed ripples or sandbars (see e.g. Davies & Heathershaw 1983, 1984; Heathershaw & Davies 1985).

The present work draws from Ehrenmark (2003) on the trapping of waves by obstacles in the presence of a plane beach. This work provided a potential function of an oscillating line source satisfying the conditions described above, and this potential is here manoeuvered into a computationally manageable Green's function whose symmetry property is rigorously demonstrated in an appendix.

The dependent variables in this problem are generally of the type $\mathrm{Re}\{(\bullet)\exp(\mathrm{i}\omega t)\}$, where ω is the circular frequency of the wave motion. They are developed in cylindrical polar coordinates, where $\theta=0$ represents the still water line (SWL); $\theta=-\alpha$ represents the flat bottom; $R=0$ is taken as the conventional shore-line; and the rigid lid of length ℓ is assumed to extend on $\theta=0$ from $R=a$ to $R=a+\ell$, where $a\geqslant 0$ can be arbitrarily varied. Here it is convenient to assume that all lengths have been non-dimensionalized by g/ω^2 and the time by ω^{-1} so that ω is explicitly removed from the problem. The primary semi-infinite domain of flow D is bounded by the SWL at the top and by the rigid bed $\theta= -\alpha$ at the bottom. The time factor and the taking of real part will usually be explicitly omitted for brevity. The further layout of the paper is as follows: Readers are first reminded of the classical solutions for linearized waves on plane beaches, developed over the years, in various forms and guises, by others but succinctly assembled and presented by Stoker (1957) in his seminal text on water waves. In that work the author first develops the two standing wave solutions mentioned earlier before going on to combine these suitably to form progressing wave solutions on a beach. The spirit of this approach is embraced in the present work, where also the standing wave components are developed first, essentially as 'building blocks' for subsequent progressing waves. The convention will be to adopt unit amplitude at infinity for the two standing wave potentials and allow radiation conditions to determine near-field behaviour.

The present author's subsequent treatment of solutions as inverse Mellin transforms is also summarized, as these forms lead to other required properties. The classical dock problem is revisited briefly, and, perhaps for the first time, solutions to this are computed fully. In § 3 are delineated the conditions required by the Green's function, and its construction is presented in the section that follows. Then, in § 5 is deduced the integral equation for the potential function which arises from application of Green's theorem in a sector to the central problem here. This involves discussing asymptotics of the classical solution with the help of which there then emerges a Fredholm equation of the second kind for the potential function on the plate. Section 6 is devoted to the numerical procedures in application and includes two validation tests for the model and some examples of application which include the cases of both (i) plate anchored at the origin (shoreline) and (ii) clear water between origin and plate (drifted plate problem).

Up to this point all solutions computed have been of the uniformly bounded type, although the development of the theory in § 5 is sufficiently general to allow description of a log-singular solution developed so that the case of progressing waves can be fully discussed. The opportunity is also taken of extension to a pair of plates. This is analysed with extensive results in § 7 showing wave envelopes disclosing enhanced standing wave behaviour shoreward of the plates.

The importance of developing a new radiation condition at the shoreline, for the beach geometry, is emphasized. Its construction (discussed from first principles in Appendix E) is achieved from the known behaviour of the fundamental standing waves, and its use means that a reflection coefficient Q for the plate in the presence of a beach can be computed in a traditional and physically meaningful way. In particular the notion of a 'transmission coefficient' is facilitated through energy flux arguments detailed in the final appendix. Sample results for Q on shallow beaches indicate excellent qualitative agreement with previous results for the finite dock problem at constant depth (Linton 2001). Concluding remarks in § 8 include ideas for further use and possible extensions of this type of model. For ease of reading, many (but not all) of the mathematical details are deferred to appendices.

2. Stoker's standing waves

In the absence of the rigid lid, the well-known wave pair $\varphi^{(r)}, \varphi^{(s)}$ which describes independent standing oscillations when $\alpha = \pi/2n$ (n integer) is fully documented first in Stoker (1947) and later in Stoker (1957, pp. 75–80). This will be used extensively in this work and is therefore recalled here for ease of reference:

$$(\varphi^{(r)}(R,\theta), \varphi^{(s)}(R,\theta)) = \mathrm{Re} \sum_{k=1}^{n} \mathrm{e}^{z\beta_k} c_k \left(1, -\mathrm{i} + \frac{1}{\pi} \int_{\mathrm{i}\infty}^{\mathrm{i}z\beta_k} \frac{\mathrm{e}^{\mathrm{i}t}}{t} \, \mathrm{d}t \right), \tag{2.1}$$

where $z = R\mathrm{e}^{\mathrm{i}\theta}$, $\beta_k = \mathrm{e}^{\mathrm{i}\pi((k/n)+(1/2))}$ and $c_k = \mathrm{e}^{\mathrm{i}\pi((n+1/4)-(k/2))} \prod_{j=1}^{k-1} \cot j\pi/2n$. The asymptotics of these potentials are as follows:

$$(\varphi^{(r)}(R,\theta), \varphi^{(s)}(R,\theta)) \sim \mathrm{e}^{R\sin\theta} \left(\cos\left[R\cos\theta + \frac{n-1}{4}\pi \right], \right.$$
$$\left. \sin\left[R\cos\theta + \frac{n-1}{4}\pi \right] \right), R \to \infty \tag{2.2}$$

and

$$\varphi^{(r)}(0,\theta) = \lim_{R\to 0} \pi\varphi^{(s)}(R,\theta)/\log R = \sqrt{k}.$$

In (2.1) the integration contour is taken anticlockwise round the origin.

The same pair was rewritten (for arbitrarily inclined beaches) as inverse Mellin transforms in Ehrenmark (1996), in the form

$$(\varphi^{(r)}(R,\theta), \varphi^{(s)}(R,\theta)) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)R^{-s}(\sin\pi s, -\cos\pi s)\,\frac{B_k(s)\cos s(\theta+\alpha)}{\sqrt{(2\pi)}\cos s\alpha}\,ds. \tag{2.3}$$

Here B_k (where now $\alpha = \pi/2k$) may be defined recursively by

$$B_k(s+N) = B_k(s+N-1)\tan\alpha(s+N-1) = B_k(s)\prod_{r=1}^{N}\tan\alpha(s+N-r) \tag{2.4}$$

and explicitly by

$$B_k(s) = \Gamma(s)\exp\left[\int_0^\infty \frac{dt}{t}\left\{\frac{2e^{t/2}\sinh(s-\frac{1}{2})t}{(e^{kt}+1)(e^t-1)} - \left(s-\frac{1}{2}\right)e^{-t}\right\}\right], \quad -k < Re\ s < k+1. \tag{2.5}$$

If k is integer then the simpler closed form is

$$B_k(s) = 2^{k-1}\sqrt{(2\pi)}\csc\pi s\prod_{j=0}^{k-1}\cos(s+j)\alpha,\ \ 0 < Re\ s < 1.$$

2.1. *The classical dock problem*

This problem, in infinitely deep water, was originally solved in Friedrichs & Lewy (1948) and represents the special case $\alpha = \pi$ (i.e. $k = 1/2$) in the above description. The contour integral solution is also described by Stoker (1957, pp. 108–109), but full computation has evidently not been described previously. To do this, one can use an integral expression for $B_k(s)$ which is easily derived from (A1.1) in Ehrenmark (1989), an equation which itself was derived from (2.5) above.

After some manipulation, there follows, on $s = (1/2) + i\tau$,

$$B_{\frac{1}{2}}(s) = \left(\frac{\pi}{\cosh\pi\tau}\right)^{\frac{1}{2}}\exp\frac{1}{2\pi i}\int_0^{2\pi\tau}\frac{x}{\sinh x}\,dx.$$

Thus we have two alternative expansions for fast computation:

$$B_{\frac{1}{2}}(s) = \left(\frac{\pi}{\cosh\pi\tau}\right)^{\frac{1}{2}}\exp i\left\{-\frac{\pi}{8} - \tau\log\tanh\pi\tau + \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{e^{-2\pi\tau(2k+1)}}{(2k+1)^2}\right\}, \tag{2.6}$$

and the small τ-expansion

$$\int_0^t \frac{x}{\sinh x}\,dx = t - \frac{t^3}{18} + \frac{7t^5}{1800} - \frac{31t^7}{105840} + \cdots \tag{2.7}$$

which can be used to six-figure accuracy up to $t = 1/2$.

One modification required for the computation arises from the preference to stick with the contour $Re(s) = 1/2$ for computational simplicity and efficiency; because the regular wave integrand has an additional simple pole at $s = 1/2$, it becomes necessary

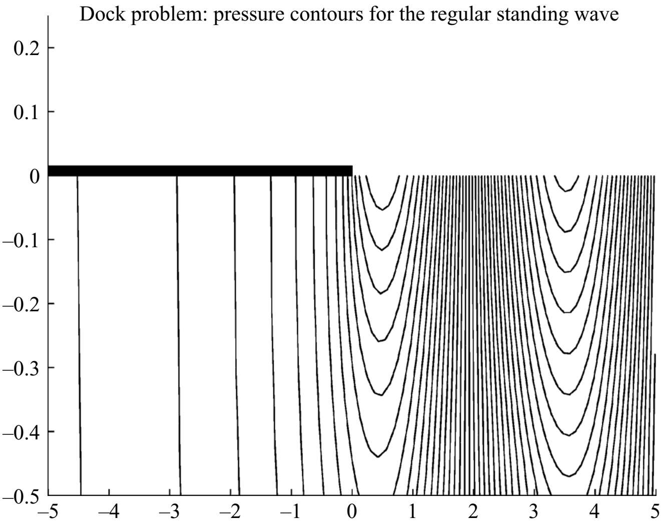

FIGURE 1. Contours calculated from an (R,θ)-grid: 50×20.

to extract half the residue there and compute instead the principal value integral. The residue is $O(R^{-1/2})$, and so this expression is not useful for small values of R. For such values it is necessary instead to shift the contour to the left over the simple pole at $s=0$ and then get back to $Re(s)=1/2$ by a mapping $s\mapsto s-1$. Thus we end up with the two alternative expressions

$$\varphi^{(r)}=\sqrt{k}+\frac{R}{2\pi\mathrm{i}\sqrt{(2\pi)}}\int_{\frac{1}{2}-\mathrm{i}\infty}^{\frac{1}{2}+\mathrm{i}\infty}\Gamma(s-1)R^{-s}B_{\frac{1}{2}}(s)\cos(s-1)(\theta+\pi)\,\mathrm{d}s;\ R<1, \tag{2.8}$$

$$\varphi^{(r)}=-\frac{\sin\frac{\theta}{2}}{\sqrt{(8\pi R)}}+\frac{1}{2\pi\mathrm{i}}v.p.\int_{\frac{1}{2}-\mathrm{i}\infty}^{\frac{1}{2}+\mathrm{i}\infty}\Gamma(s)R^{-s}\sin\pi s\frac{B_{\frac{1}{2}}(s)\cos s(\theta+\pi)}{\sqrt{(2\pi)}\cos s\pi}\,\mathrm{d}s,\ \ R\geqslant 1. \tag{2.9}$$

The reader will want to note that the fundamental initial solution contour cannot be to the right of the pole at $s=1/2$, hence the minus sign on the residue contribution. Some further details are delivered in Appendix C.

Pressure contours are shown in figures 1 and 2 for the regular and singular standing waves.

3. The development of a Green's function integral

Although the rigid lid boundary condition will be generalized, use will be made of a Green's function which satisfies the conventional free surface condition on the SWL. The requirements of $G(\underline{R}|\underline{z})$, where $\underline{R}$ denotes the field point and $\underline{z}$ denotes the source point (expressed in complex notation as $\zeta=R\mathrm{e}^{\mathrm{i}\theta}$, $z=\rho\mathrm{e}^{\mathrm{i}\gamma}$ respectively), are

$$R^{-1}G_{\theta}(R,0|\underline{z})=G(R,0|\underline{z}), \tag{3.1}$$

$$G_{\theta}(R,-\alpha|\underline{z})=0, \tag{3.2}$$

$$\Delta G(R,\theta|\underline{z})=2\pi\delta(\underline{R}-\underline{z}),\quad \underline{R}\in D, \tag{3.3}$$

in addition to which we take suitable boundedness requirements at infinity and the shoreline. Thus $G(\underline{R},\underline{z})$ will effectively be the potential function applicable when an oscillating point source is placed at $\underline{z}$ in the presence of the bed $\theta=-\alpha$. Such a

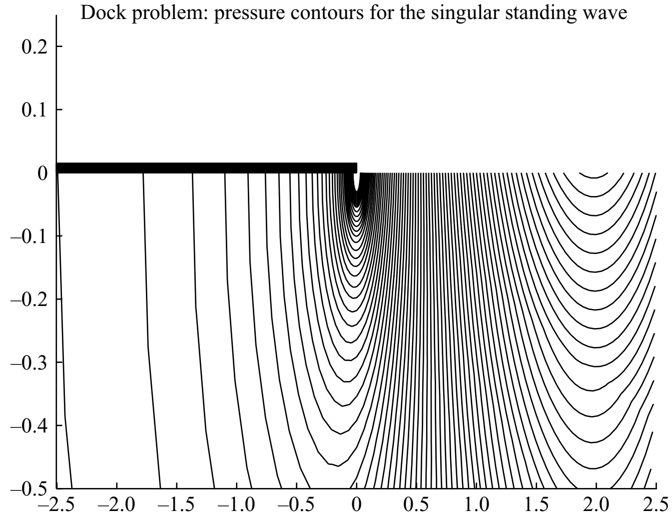

FIGURE 2. Contours calculated from an (R, θ)-grid: 50×20.

potential was written by the author in a recent work Ehrenmark (2003) and will be used in the following section.

The physical problem to be considered will be described by a velocity potential ϕ satisfying the following:

$$\Delta\phi(\underline{R}) = 0, \quad \underline{R} \in D, \tag{3.4}$$

$$\phi_\theta(R, -\alpha) = 0, \quad R \in (0, \infty), \tag{3.5}$$

$$R^{-1}\phi_\theta(R, 0) = \lambda\phi(R, 0) \quad R \in (a, a+\ell),\ 0 \leqslant \lambda \leqslant 1, \tag{3.6}$$

$$R^{-1}\phi_\theta(R, 0) = \phi(R, 0) \quad R \in (0, a) \cup (a+\ell, \infty). \tag{3.7}$$

If the reflection is perfect, it can assumed that the amplitude of ϕ as $R \to \infty$, (ϕ_∞), is known but not its phase. However, the generalization is made to include the wave which is logarithmically unbounded at the origin.

The parameter λ allows (i) consideration of the rigid lid problem ($\lambda = 0$), (ii) permeability in the lid $0 < \lambda < 1$ and (iii) recovery of the classical solution as a check ($\lambda = 1$). Then from the usual use of the Green's identity, the solution may be formally written in the form

$$\phi(\underline{z}) = \frac{1}{2\pi}\int_{\partial D} (\phi(\underline{R})G_n(\underline{R}, \underline{z}) - \phi_n(\underline{R})G(\underline{R}, \underline{z}))\,\mathrm{d}s, \quad \underline{R} \in \partial D, \tag{3.8}$$

where n indicates a normal direction drawn out of D and ∂D is described counter-clockwise. It is noted that ϕ remains finite as $R \to \infty$ on the surface. Let L_{R_0} be a radial arc of sufficiently large radius (R_0). It follows that we can take for ∂D the union of the large arc, the mean free surface $\theta = 0$ and a small arc L_0 drawn to exclude a weak singularity of ϕ arising at $\underline{R} = 0$. It is anticipated that the solution will be continuous at the tip(s) of the plate but that its gradient will be logarithmically singular there (see Linton 2001).

4. Construction of a Green's function

The two-dimensional problem (i.e. without long-shore variation) of a submerged pulsating source (and also dipole) placed above a flat beach of arbitrary slope was solved in Ehrenmark (2003) in conjunction with an investigation of wave trapping by obstacles. With the source placed at $z=\rho e^{i\gamma}$ or (ρ,γ) in polar coordinates, an expression providing a potential function which is uniformly bounded in any domain with a δ-disk removed at the source, may be written (see (7.1)–(7.4) in Ehrenmark 2003)

$$G(R,\theta|\rho,\gamma)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}R^{-s}B_k(s)\sin\pi s\ \Gamma(s)d(s|\rho,\gamma)\frac{\cos s(\theta+\alpha)}{\cos s\alpha}\,ds+G^0(\zeta|z) \tag{4.1}$$

where $\zeta=Re^{i\theta}$,

$$G^0(\zeta|z)=\log\left|\frac{(\zeta^{\pi/\alpha}-z^{\pi/\alpha})(\zeta^{\pi/\alpha}-\overline{z}^{\ \pi/\alpha})}{\zeta^{2\pi/\alpha}}\right|, \tag{4.2}$$

and a particular integral d_1, of the original difference equation (from Appendix A and Ehrenmark 2003 or using a Mellin transform), is

$$d_1(s|\rho,\gamma)=-\pi i\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\cos\tau(\gamma+\alpha)\rho^{\tau}\cot\pi(s-\tau)}{\sin\tau\alpha\ \Gamma(\tau+1)B_k(\tau)\sin\tau\pi}\,d\tau,$$

where $0<\sigma<1$ and where we choose the branch for which $\sigma-1<Re\ s<\sigma$. The integral converges (in the sense of Cauchy) in this interval, and the convergence is uniform in $-\alpha\leqslant\gamma\leqslant 0$, for all $\alpha\leqslant\pi$. We need to add a suitable 'complementary function' (i.e. one that is periodic with real period unity) in order to remove the log singularity in $G^{(0)}$ at $R=0$. The required solution d is therefore given by

$$d(s|\rho,\gamma)=-\frac{2\pi}{\sqrt{\alpha}}\cot\pi s-\pi i\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\cos\tau(\gamma+\alpha)\rho^{\tau}\cot\pi(s-\tau)}{\sin\tau\alpha\ \Gamma(\tau+1)B_k(\tau)\sin\tau\pi}\,d\tau. \tag{4.3}$$

The integral here is analytic in the strip $-1<Re\ s<1$, so the first term determines the primary behaviour of d near $s=0$. Note that $\partial G^0/\partial\theta=0$ on $\theta=-\alpha,0$ and also that $\partial G^0/\partial\gamma=0$ on $\gamma=-\alpha,0$. Note further that if k is an integer (unless explicitly stated otherwise, this assumption will be retained for convenience), then $1/(B_k(s)\sin s\alpha)$ is regular in the entire right-hand half-plane. The residue theorem may be used in this plane to give a full expansion.

Write

$$f_k(s|\rho)=\rho^s\sum_{N=1}^{\infty}\frac{(-\rho)^N\cos(s+N)(\gamma+\alpha)}{\sin(s+N)\alpha\ \Gamma(1+s+N)B_k(s+N)}.$$

There then follows, using the residue theorem in (4.3),

$$d(s|\rho,\gamma)\sin s\pi=2\pi\left(-\frac{\cos s\pi}{\sqrt{\alpha}}+f_k(s|\rho)-f_k(0|\rho)\cos s\pi\right). \tag{4.4}$$

By substitution in (4.1) the Green's function may now be expressed in the form

$$G(\zeta|z)=G^0(\zeta|z)+(2\pi)^{3/2}\left(\frac{1}{\sqrt{\alpha}}+f_k(0|\rho)\right)\varphi^{(s)}(R,\theta)+\frac{1}{i}\int_{c-i\infty}^{c+i\infty}R^{-s}B_k(s)$$
$$\times\ \Gamma(s)f_k(s|\rho)\frac{\cos s(\theta+\alpha)}{\cos s\alpha}\,ds.$$

Here, $\varphi^{(s)}(R,\theta)$ is the 'singular' standing wave of the classical scattering problem defined first by Stoker (1947) in terms of exponential integrals and here as (2.3) and also in Ehrenmark (1996) as a near-field expansion in the form

$$\varphi^{(s)}(R,\theta)=\frac{\sqrt{k}}{\pi}\sum_{N=0}^{\infty}\frac{\rho_N R^N}{N!}(-\lambda_N+\log R), \tag{4.5}$$

where, if $0\leqslant N<k$,

$$\lambda_N=-\alpha\sum_{j=1}^{k-1}\tan j\alpha+\psi(1+N)+2\alpha\sum_{j=1}^{N}\frac{1}{\sin 2j\alpha}+\Theta\tan N\Theta-\alpha\tan N\alpha,\qquad \Theta=\theta+\alpha$$

and

$$\rho_N=\frac{\cos N\Theta}{\cos N\alpha}\prod_{j=1}^{N}(-\cot j\alpha),\ N>0;\ \ \rho_0=1;\ \ \rho_k=(-)^k\cos k\Theta,$$

it being understood that $\Sigma_{j=1}^{N}$ is null if $N=0$. Note that, in the above, care has to be taken to ensure cancelation of zero and pole each time N is an integer multiple of k. For that reason, it may be preferable to use directly the full integral expression to compute G by substituting (4.4). The advantage with the expansion approach is that logarithmic terms near $R=0$ are eliminated exactly instead of numerically.

For the evaluation of $f_k(0|\rho)$, using the recurrence relation for $B(s)$ and the value $B(1)=\sqrt{\alpha}$, one can write

$$\frac{1}{\sqrt{\alpha}}+f_k(0|\rho)=\frac{1}{\sqrt{\alpha}}\sum_{N=0}^{\infty}\frac{\rho^N}{N!}\,d_N\cos N(\gamma+\alpha),$$

where

$$d_N=\frac{1}{\cos N\alpha}\prod_{j=1}^{N}-\cot j\alpha;\ \ d_0=1.$$

The expression is now seen to be a fixed multiple $\sqrt{(2/\pi)}$ of $\varphi^{(r)}(\rho,\gamma)$ the regular standing wave for the classical scattering problem given in Stoker (1947) in closed form and in Ehrenmark (1996) as the above expansion. For consistency with Ehrenmark (1996) – apart from a sign change – $\varphi^{(r)}(\rho,\gamma)$ is chosen (like $\varphi^{(s)}(R,\theta)$) to have unit amplitude at infinity with $\varphi^{(s)}(R,\theta)\sim\varphi_\infty^{(s)}(R,\theta)=-\cos\chi\,\mathrm{e}^{R\sin\theta}$ and $\varphi^{(r)}(R,\theta)\sim\varphi_\infty^{(r)}=\sin\chi\,\mathrm{e}^{R\sin\theta}$, where $\chi=R\cos\theta+\pi(1+k)/4$. Then also $\varphi^{(r)}(0,\theta)=\sqrt{k}$, whilst $\varphi^{(s)}(R,\theta)/\log R\to\sqrt{k}/\pi$ as $R\to 0$. There then follows a simplified expression for the Green's function

$$G(\zeta|z)=G^0(\zeta|z)+4\pi\varphi^{(r)}(\rho,\gamma)\varphi^{(s)}(R,\theta)+\frac{1}{\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}R^{-s}B(s)$$
$$\times\ \Gamma(s)f_k(s|\rho)\frac{\cos s(\theta+\alpha)}{\cos s\alpha}\,\mathrm{d}s. \tag{4.6}$$

Note that, with $c=1/2$ this last integral is absolutely convergent, at worst $O(s^{-2})$ if both source and field points are on the surface and otherwise exponentially convergent. For non-integer k the summation inside the integral needs to be replaced by a further integral, but if k is an integer, for numerical quadrature, the integrand is best expressed in the form

$$\frac{\cos s(\theta+\alpha)}{\cos s\alpha}\left(\frac{R}{\rho}\right)^{-s}\sum_{N=1}^{\infty}\frac{\cos(s+N)(\gamma+\alpha)}{(s+N)\sin(s+N)\alpha}\prod_{j=0}^{N-1}\frac{(-\rho)\cot(s+j)\alpha}{s+j}.$$

The expression (4.6) is readily evaluated for all field points, but for very small values of R there is the numerical inconvenience of logarithmically large terms cancelling out. To avoid this, the symmetry property $G(\zeta|z) = G(z|\zeta)$ (established in Appendix B) is used. This provides representation for small field points and in particular, by demonstrating that $d(s)$ is regular at $s = -1$, it follows after calculating a residue that

$$G(\epsilon, \theta|\underline{z}) = 4\pi\sqrt{k}\varphi^{(s)}(\rho, \gamma)\left(1 + \frac{\epsilon\sqrt{k}}{\pi}\frac{\cos(\theta + \alpha)}{\sin\alpha}\right) + O(\epsilon^2).$$

In much the same light, (4.6) provides the expected result $G(\zeta|z) \to 4\pi\sqrt{k}\ \varphi^{(s)}(R, \theta)$ as $\underline{z} \to 0$ (i.e. the source is placed at the origin). One would naturally expect this placement to result in G being identified with Stoker's fundamental singular wave. Both these results hinge on the observation that the respective 'remainder integrals' can be summed by the residue theorem to provide convergent expansions in only positive powers of respectively R (contour completed to the left) and ρ (contour completed to the right).

From (4.6) we deduce also the large R asymptotics,

$$G(\zeta|z) \to 4\pi\varphi^{(r)}(\rho, \gamma)\varphi_\infty^{(s)}(R, \theta), \quad R \to \infty.$$

Prior to more general application, it is of interest to consider next the special case of $\alpha = \pi$, as it will be used later as a validation tool in connection with the regular wave.

4.1. *The case of the dock:* $\alpha = \pi$

The fundamental solutions to the dock problem were discussed earlier, and if, for example, we take a plate of length unity anchored at the origin, then we effectively get the same dock problem with the origin moved one unit to the left.

One complication is that the specific form for the Green's function used in (4.6) will need to be replaced by one arising from the direct use of formula (4.3) in (4.1) so that the defining integral becomes

$$G(R, \theta|\rho, \gamma) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} R^{-s}B_k(s+1)\ \Gamma(s)d(s|\rho, \gamma)\cos s(\theta+\pi)\,\mathrm{d}s + G^0(\zeta|z), \qquad (4.7)$$

where now

$$d(s|\rho, \gamma) = -2\sqrt{\pi}\cot\pi s - \frac{\rho}{i\pi}\int_{c-i\infty}^{c+i\infty}\frac{\cos(1-\tau)(\gamma+\pi)\rho^{-\tau}\cot\pi(s+\tau)\Gamma(\tau)B_k(\tau)}{\tau - 1}\,\mathrm{d}\tau, \qquad (4.8)$$

following some further manipulation and use of the folding formulae. Although values of G will only be required on $\theta = 0$ it is convenient to retain the general form, as this will permit exploitation of the symmetry (see Appendix C) in cases in which $\rho < R$. In order to proceed with a balanced approach to the iterated integration, it is convenient to compute $\Gamma(s)B_{1/2}(s)\sin\pi s$ on $s = (1/4) + iy$ to leave the remainder of the integrands also without exponential decay or growth. That then leaves two weakly converging oscillatory integrals which are readily computed using Sidi's W-transformation (see e.g. Sidi 1988), a technique which is particularly useful for slowly converging such integrals. Note that the first term of (4.8) provides a term $G_1 = 4\pi\sqrt{k}\varphi_s(R, \theta)$ when inserted into (4.7).

5. Green's theorem

The behaviour of ϕ at infinity means that we need to take a finite region D bounded by the large arc $R = R_0$, the bed and the SWL. In the interest of generality, subsequently to account also for the singular solution, we anticipate a possible log singularity at $R = 0$, and this would necessitate that point to be excluded from D by means of a small circular arc of radius ϵ. Green's theorem is then taken in the fundamental form

$$\phi(z) = \frac{1}{2\pi}\int_{\partial D}\{\phi(\zeta)G_n(\zeta|z) - G(\zeta|z)\phi_n(\zeta)\}\,\mathrm{d}s, \tag{5.1}$$

where n is directed out of D and ∂D is traversed anticlockwise.

In order to develop the above into an integral equation for values of ϕ on the plate, it will be necessary to expand ϕ and $\varphi^{(r,s)}$ on both of the two arcs. On the large arc (as $R_0 \to \infty$), we take $\phi \sim A\ \varphi^{(r)}_\infty + B\varphi^{(s)}_\infty$, whilst on the smaller arc (as $\epsilon \to 0$) we posit

$$\phi \sim c\log\epsilon + O(1)$$

for suitably determined complex constants A, B, c. The value $c = 0$ leads to a perfectly reflected bounded wave with the plate controlling only phase shift. Whilst convenient as illustrator of the method of solution (see figures 3–9), the assumption that plate and beach together form a perfect reflector is somewhat artificial, and a more realistic approach perhaps is to develop a radiation condition at the origin which allows energy to escape there but not to emanate from there. This is done in § 7, where progressing waves are considered more generally. Details of this new radiation condition are developed from first principles in Appendix E.

5.1. *The arc* $\epsilon \to 0$

The contribution to the right-hand side of (5.1) from this arc is given by

$$-\lim_{\epsilon\to 0}\frac{\epsilon}{2\pi}\int_{-\alpha}^{0}\left\{\phi(\epsilon,\theta)\frac{\partial G(\epsilon,\theta|\underline{z})}{\partial R} - G(\epsilon,\theta|\underline{z})\frac{\partial\phi(\epsilon,\theta)}{\partial R}\right\}\mathrm{d}\theta = 2c\alpha\sqrt{k}\varphi^{(s)}(\underline{z})$$

since $\partial G(\epsilon,\theta|\underline{z})/\partial R = O(1),\ \epsilon \to 0$.

5.2. *The arc* $R_0 \to \infty$

The required expression here is

$$I = \lim_{R_0\to\infty}\frac{R_0}{2\pi}\int_{-\alpha}^{0}\left\{\phi(R_0,\theta)\frac{\partial G}{\partial R}(R_0,\theta|\underline{z}) - G(R_0,\theta|\underline{z})\frac{\partial\phi}{\partial R}(R_0,\theta)\right\}\mathrm{d}\theta.$$

From (4.6) it follows that

$$\lim_{R_0\to\infty} G(R_0,\theta|\underline{z}) \to 4\pi\varphi^{(r)}(\underline{z})\varphi^{(s)}(R_0,\theta);$$

by inserting this and the asymptotics for ϕ assumed above, there follows

$$I = 2R_0A\varphi^{(r)}(\underline{z})\int_{-\alpha}^{0}\left\{\varphi^{(r)}_\infty\frac{\partial\varphi^{(s)}_\infty}{\partial R} - \varphi^{(s)}_\infty\frac{\partial\varphi^{(r)}_\infty}{\partial R}\right\}\mathrm{d}\theta.$$

Use of the asymptotic forms for $\varphi^{(r,s)}$ (2.2) enables the quadrature to be completed, and we get

$$I = A\ \varphi^{(r)}(\underline{z}).$$

5.3. *The integral equation*

Green's theorem then expands to

$$\phi(\underline{z}) = A\ \varphi^{(r)}(\underline{z}) + \frac{1-\lambda}{2\pi}\int_a^{a+\ell}\phi(R,0)G(R,0|\underline{z})\,\mathrm{d}R + 2c\alpha\sqrt{k}\varphi^{(s)}(\underline{z}). \tag{5.2}$$

If we let $z \to \infty$ in this equation we obtain

$$B = 2(1-\lambda)\int_a^{a+\ell}\phi(R,0)\varphi^{(r)}(R,0)\,\mathrm{d}R + 2c\alpha\sqrt{k}. \tag{5.3}$$

Similarly if we let $z \to 0$ we obtain, purely as a check, $\phi(\rho,\gamma)\sim c\log\rho$ as $\rho \to 0$.

Then (5.3), which can also be derived by applying Green's formula to ϕ and $\varphi^{(r)}$, is complemented by a radiation condition to determine the field at infinity. The equation for ϕ becomes

$$\phi(\underline{z}) = A\varphi^{(r)}(\underline{z}) + T\varphi^{(s)}(\underline{z}) + \frac{1-\lambda}{2\pi}\int_a^{a+\ell}\phi(R,0)G(R,0|\underline{z})\,\mathrm{d}R \tag{5.4}$$

after eliminating c using (5.3). Here

$$T = c\pi/\sqrt{k} = B - 2(1-\lambda)\int_a^{a+\ell}\varphi^{(r)}(x,0)\phi(x,0)\,\mathrm{d}x. \tag{5.5}$$

Now select $z=\xi$, where $a\leqslant\xi\leqslant a+\ell$, and there follows an integral equation for $\phi(\xi,0)$ namely

$$\phi(\xi,0) = A\varphi^{(r)}(\xi,0) + T\varphi^{(s)}(\xi,0) + \frac{(1-\lambda)}{2\pi}\int_a^{a+\ell}\phi(R,0)G(R,0|\xi,0)\,\mathrm{d}R. \tag{5.6}$$

Next, it is convenient to write

$$\phi^{(r,s)}(\xi) = \varphi^{(r,s)}(\xi,0) + \frac{(1-\lambda)}{2\pi}\int_a^{a+\ell}\phi^{(r,s)}(R)G(R,0|\xi,0)\,\mathrm{d}R \tag{5.7}$$

or, in operator form,

$$(I-K)\phi^{(r,s)} = \varphi^{(r,s)}, \tag{5.8}$$

where

$$(K\psi)(\xi) = \frac{(1-\lambda)}{2\pi}\int_a^{a+\ell}\psi(R)G(R,0|\xi,0)\,\mathrm{d}R.$$

It is readily seen that K is a self-adjoint compact operator on the Hilbert space $L_2(a,a+\ell)$. The question of uniqueness addresses the issue of inverting the operator $(I-K)$, and although a theoretical investigation appears to be out of reach it will be seen later that the numerical approximation to $(I-K)$ is invertible for an unspecified range of parameter values. Thus a quest for eigenvalues of the homogeneous problem is not undertaken here. (Note that either a, λ or ℓ could be treated as the eigenvalue parameter here.)

By selecting z to be on the plate in (5.4), there follows

$$\phi(\xi,0) = A\phi^{(r)}(\xi) + T\phi^{(s)}(\xi). \tag{5.9}$$

Substitution of this into (5.5) leads to

$$\chi_0 T = B - 2A(1-\lambda)\int_a^{a+\ell}\varphi^{(r)}(R,0)\phi^{(r)}(R)\,\mathrm{d}R, \tag{5.10}$$

where

$$\chi_0 = 1 + 2(1-\lambda)\int_a^{a+\ell} \varphi^{(r)}(R,0)\phi^{(s)}(R)\,\mathrm{d}R.$$

Corollary: Note that, for a wave bounded at the shore (perfectly reflected wave), we choose $T=0$ (i.e. $c=0$); so then, with $A=\cos\beta,\ B=\sin\beta$, the phase shift β is determined by

$$\tan\beta = 2(1-\lambda)\int_a^{a+\ell} \varphi^{(r)}(R,0)\phi^{(r)}(R)\,\mathrm{d}R. \tag{5.11}$$

We begin the numerical procedures below though by first computing this bounded standing wave in various situations including the case in which the plate is attached (anchored) at the shoreline. The computation of progressing waves is deferred to § 7, where also the construction of appropriate radiation conditions are discussed.

6. Numerical procedure for the rigid platform problem

Although the case of perfect reflection is somewhat artificial physically, it is nevertheless the simplest and is thus well suited for discussing the numerical techniques; moreover it also provides suitable validation tests.

For the case of the bounded wave, approximations are required to satisfy the equation

$$\phi(\rho,0) = \cos\beta\varphi^{(r)}(\rho,0) + \frac{1-\lambda}{2\pi}\int_a^{a+\ell} G(R,0|\rho,0)\phi(R,0)\,\mathrm{d}R,\ \ a\leqslant\rho\leqslant a+\ell. \tag{6.1}$$

By writing, for convenience, $\phi^{(r)}(R)\equiv\psi(R)$ and substituting

$$\phi(R,0) = \cos\beta\cdot\psi(R),$$

provided $\cos\beta\neq 0$, the problem to determine ϕ on the plate can be written as

$$\psi(\rho) = f(\rho) + \kappa\int_a^{a+\ell} G(R|\rho)\psi(R)\,\mathrm{d}R, \tag{6.2}$$

where $\kappa=(1-\lambda)/2\pi$; $f(\rho)=\varphi^{(r)}(\rho,0)$; and it is understood for this that G is computed with $\theta=\gamma=0$. The consistency requirement described earlier has now been decoupled from the system so that β may be determined from

$$\tan\beta = 4\pi\kappa\int_a^{a+\ell} f(R)\psi(R)\,\mathrm{d}R. \tag{6.3}$$

We denote a set of abscissae by $x=x_i=a+(i-1)h,\ i=1,2,\ldots,N+1, Nh=\ell$ and the approximations corresponding to ψ by v_i. The successive application of a suitable quadrature in (6.2) will then yield a linear system $v_i=b_{ij}v_j+f(x_i)$ (where b_{ij} are a set of coefficients dependent on the Green's function and the particular quadrature rule), from which the approximations are readily determined numerically. The symmetry of the Green's function (see Appendix B) will ensure the essential symmetry of the b_{ij}. The use will be made of a trapezoidal rule application suitably modified (see below) to accommodate the log singularity when field point and source point are both on the plate. Details of the method are given in Appendix D.

6.1. *The case* $a=0$

The integration is on $[0,\ell]$, and the approximation to G on the first strip of the trapezoidal rule can be evaluated as above with the correction found from the

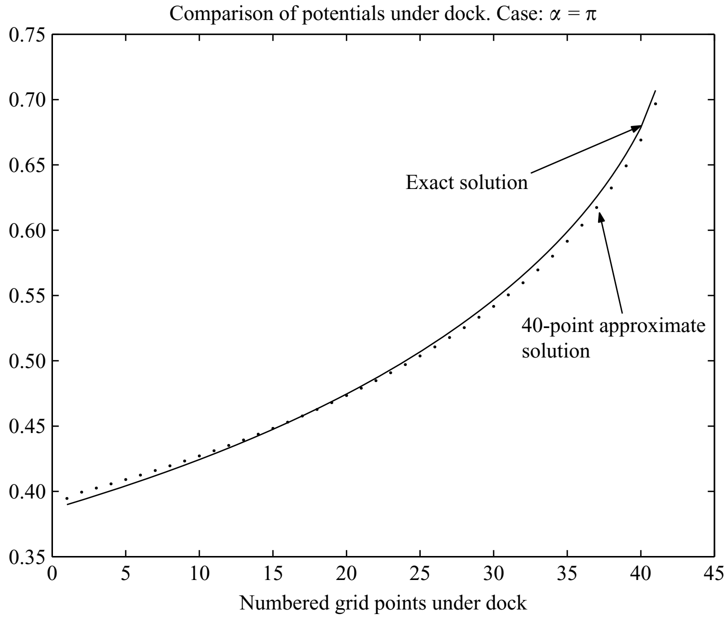

FIGURE 3. Validation of model using the semi-infinite dock classical solution.

expansion of $\varphi^{(s)}$ given in Ehrenmark (1996, p. 123). Therein, it is found that $\varphi^{(s)} = (\sqrt{k}/\pi)(\log R - \Psi(1) - B'(1)/B(1)) + b_1 R \log R + O(R)$, where b_1 can be found and $-\Psi(1) \approx 0.577216$ is Euler's constant. The evaluation of $B'(1)$ would require numerical quadrature except for the very simple beach angles at which where k is integer or 1/2, whence $B'(1)/B(1) = -\alpha \sum_{j=1}^{k-1} \tan j\alpha$, taking the value 0 when $k = 1$ (the vertical cliff case) and also 1 when $k = 1/2$, (the dock problem; see Appendix C). In order to have the numerical procedure with error $O(h^2)$, it is necessary to include also the term $b_1(R \log R)$, but this is messy to evaluate and can instead (along with any further terms required) be determined numerically, since the precise nature of the expansion is known (Ehrenmark 1996).

6.1.1. *Validation 1*

The case of the dock (i.e. $\alpha = \pi$ or $k = 1/2$) is considered first, as it provides a means of validating the model robustly. The far-field asymptotics of the classical solution (given earlier) show that the value of β recovered in (6.3) should be $-\ell$ when this particular model is run.

With no loss of generality, $\ell = 1$ may be specified. Then the value $\beta = -0.9948$ is achieved with a 40-point discretization on the plate. The comparison is shown in figure 3 between the potentials calculated immediately under the plate by the discretization and the exact solution calculated from equations in § 2.1. The numerical integration has been subject to some refinement near the log-singular points. As pointed out by Hough (1982) accurate integration with log-singular kernels of this type may benefit from distinguishing between 'field points' both near and distant from the source point. A sensible balance then has to be found between the labour of doing additional 'exact' log-term integration and the benefit accruing from it.

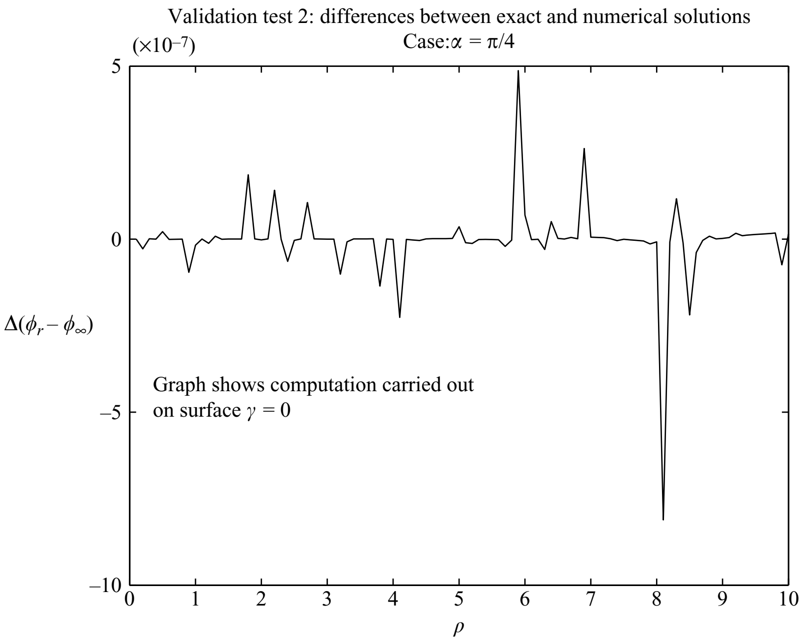

FIGURE 4. Validation of model by setting $\ell = 0$.

6.1.2. *Validation 2*

A second simple validation procedure for the Green's function is given by setting $\ell = 0$. This provides the classical solution which, for the case $\alpha = \pi/4$, for example, can be written

$$\varphi_r = \phi_\infty + \mathrm{e}^{-x}(\cos y + \sin y)/\sqrt{2}$$

in Cartesians when

$$\phi_\infty = \mathrm{e}^{y}(\cos x - \sin x)/\sqrt{2}.$$

Application of Green's theorem to $\varphi - \phi_\infty$ then yields

$$(\varphi_r - \phi_\infty)(\underline{z}) = \frac{1}{2\pi}\int_{R=0}^{\infty} G(R, -\alpha|\underline{z})\mathrm{e}^{-R/\sqrt{2}} \sin(R/\sqrt{2})\,\mathrm{d}R.$$

Computation of this, on a grid $\rho \times \gamma = [0, 0.1, 10] \times [0, 0]$, is compared with the exact solution by calculation of their difference $\Delta(\varphi_r - \phi_\infty)$ for this beach angle. Results are shown in figure 4.

6.1.3. *Application with beach* $\alpha = \pi/4$

In this case $k = 2$, and so k being integer, the simpler numerical strategy can be used for the Green's function. Although this does not greatly affect solving for ϕ on the plate, it significantly reduces the subsequent computing time for evaluation of ϕ at a large number of points in the water column. The integration is achieved with $N = 100$, i.e. $h = 0.01$. It is readily observed (see figure 5) that the zone under the plate is comparatively quiescent. The value of $\delta\phi/\delta y$ has also been computed at all surface nodes except the tip at $x = 1$. Note also from the diagram how these approximate well to $\partial\phi/\partial y = 0$ on $0 < x < 1$ and to $\partial\phi/\partial y = \phi$ on $x > 1$. This provides a further validation of the model.

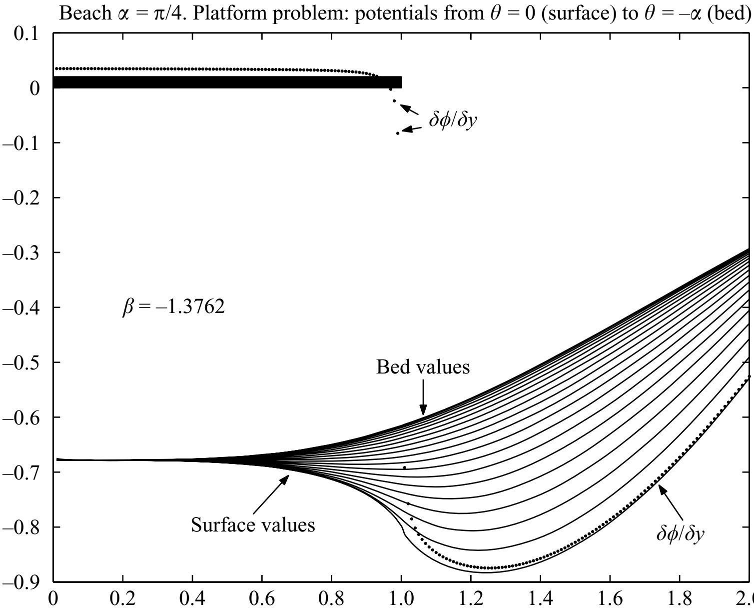

FIGURE 5. Water column potentials under and beyond a unit length plate anchored at the origin. Note the computation of the finite difference approximation to $\partial\phi/\partial y$ with $\delta y = h = 0.01$.

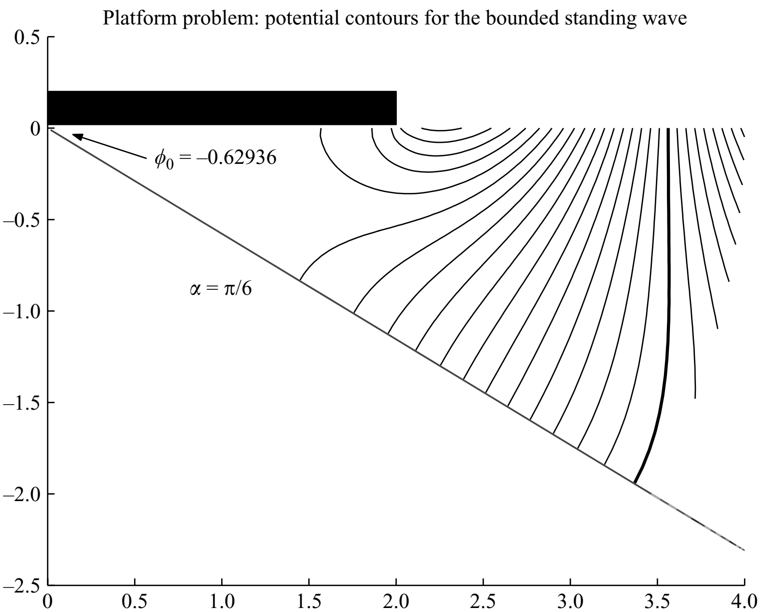

FIGURE 6. Equipotentials for the anchored plate problem on a 30° beach. Plate anchored on [0, 2]; thicker line is zero potential.

6.1.4. *Application with beach* $\alpha = \pi/6$

For this case, equipotential contours are instead presented in figure 6 in the range $\phi = [-1.1 : 0.05 : 1]$. Note the extent of the 'quiescent' region under the plate.

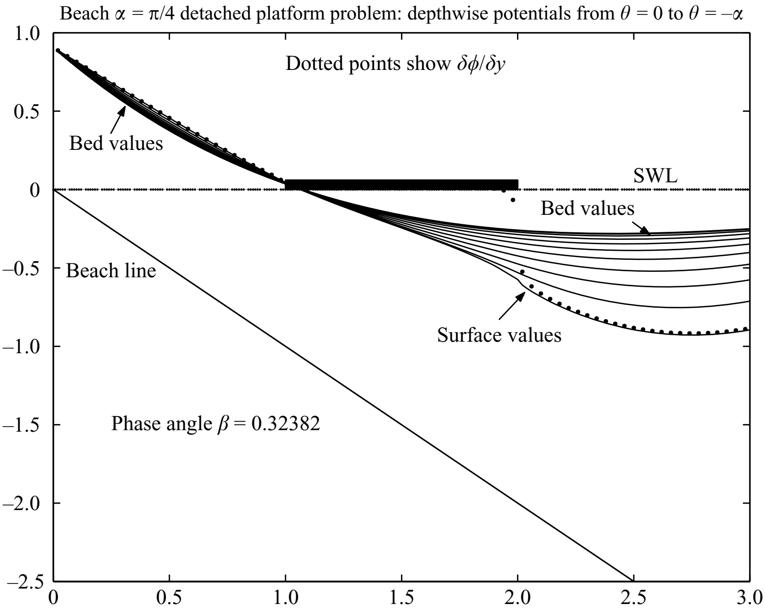

FIGURE 7. Water-column potentials for the drifting plate problem on a 45° beach.

6.2. *The case $a \neq 0$*

In this case there is clear water between the origin and the plate. The effect of this is to mitigate part of the singularity that has to be accommodated on the first element of the discretization. The singular contribution from $\varphi^{(s)}$ is lost, and only the term arising from $G^{(0)}$ need be given the special treatment. This leads to precise symmetry of the matrix b_{ij}.

For the case in which the plate occupies $1 \leqslant x \leqslant 2$, the water column potentials are shown for 10 depths in figure 7, whilst the potential contours are displayed in figure 8. Note again, in figure 7 how the approximate value of $\partial\phi/\partial y$ follows closely that required by the surface boundary conditions. A further application for the beach angle $\alpha = \pi/6$ is displayed in figure 9, where water column potentials are plotted alongside the fundamental wave $\varphi^{(r)}$ that would represent the bounded wave in the absence of the plate. This shows more clearly the discontinuity on the surface, in $\partial\phi/\partial R$ at the plate tips.

7. Progressing waves and a shoreline radiation condition

7.1. *Constructing a progressing wave*

As discussed earlier, the formation of a progressing wave can be achieved from two fundamental standing waves which are out of phase by $\pi/2$ at infinity. If these are denoted Φ_r and Φ_s, then a pure progressing wave travelling towards shore may be given by

$$\Phi = Re\{(\Phi_r + \mathrm{i}\Phi_s)\mathrm{e}^{\mathrm{i}t}\}.$$

Similarly a wave travelling to the right is obtained by sign change from + to − in the above. In the absence of a plate (the classical beach problem) the usual choice is $\beta = 0$ and $\beta = \pi/2$ respectively, thus providing $\Phi_{(r,s)} = \varphi^{(r,s)}$ in this case. This wave is therefore logarithmically singular at the shore.

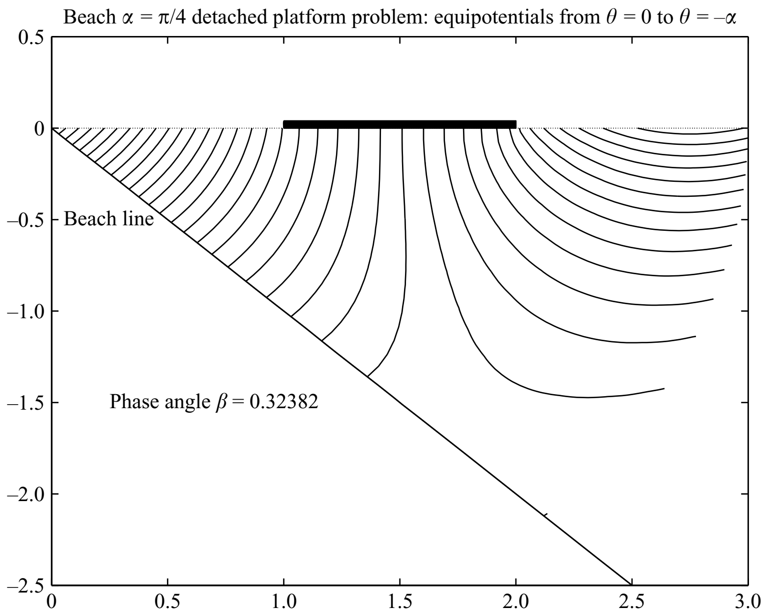

FIGURE 8. Equipotentials for the drifting plate problem on a 45° beach.

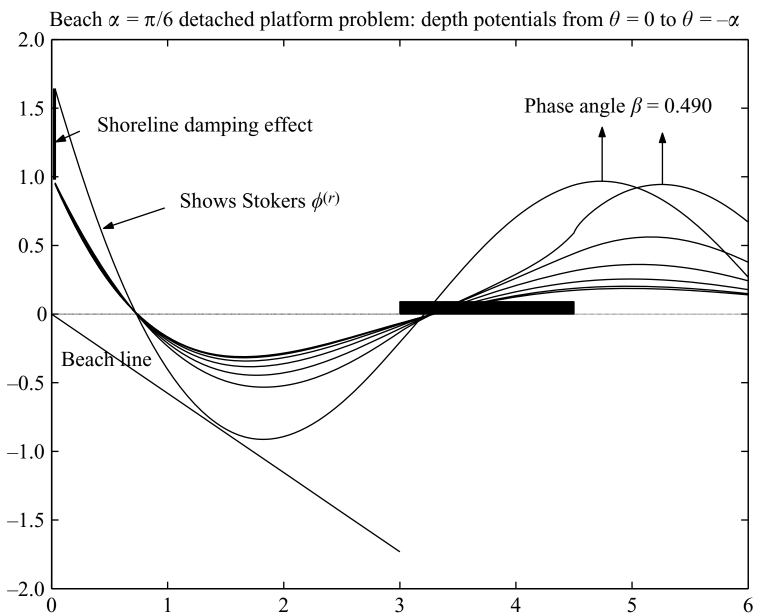

FIGURE 9. Water column potentials for the bounded-wave drifting-plate problem on a 30° beach. Plate end point coordinates: 3.0 and 4.5. Note effective shoreline damping for this configuration.

In the presence of the plate, we could adopt the same approach (with the choice of β), but then neither of the two constituents $\Phi_{(r,s)}$ would be bounded at the shore. Alternatively, we can see that if we choose $\Phi_{(r)}$ by setting $I=0$ and then $\Phi_{(s)}$ by setting $A=0$, we will instead get two standing waves, where only the first one is

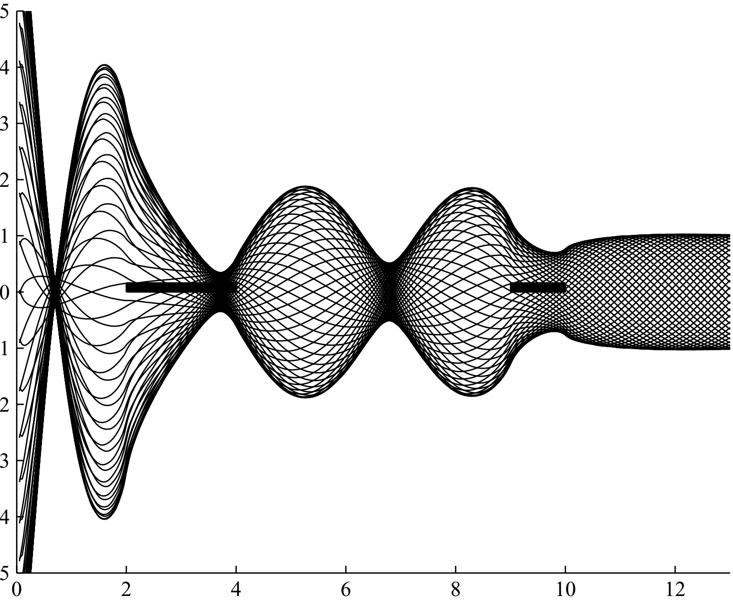

FIGURE 10. Wave envelope for the case of pure incoming progressive wave on two plates from ∞ with the wide plate nearer the shore; beach angle $= \pi/8$.

bounded at the shore but where they now have a phase difference other than $\pi/2$ at infinity. This is essentially the approach adopted in order to accommodate the treatment of the diffraction problem, using radiation conditions.

7.2. *Computation of progressing waves*

By computing each of the two constituents Φ_r, Φ_s separately, wave envelopes are readily obtained. This can be done also for the case of the two-plate problem as described in some detail in § D.1 of Appendix D. Notwithstanding the logarithmic singularity at the shore which technically implies standing wave dominance there, the results shown in figures 10 and 11 clearly indicate a tendency for standing waves to develop more generally shoreward of the obstructing plate(s). The plate positions are indicated in these diagrams, but the beach bed is not shown. In figures 10 and 11 the computations were made on a beach of slope $\pi/8$. Here the value of c is non-zero, and energy is being pumped into the fluid at $+\infty$ and out at the shoreline. However, it needs to be understood that the plates will have created some reflective component R_0, but as this has been removed by the stipulation that there is no reflected wave at infinity, it follows that there is a mitigating negative right-travelling wave emanating from the origin to cancel R_0.

Examples of waves reflected by plates are well documented for cases of constant depth (see e.g. Linton 2001; Hermans 2003), where solutions are obtained through the application of appropriate radiation conditions, normally at $x = \pm\infty$. For the beach configuration, the present author is unaware of the previous use, in a non-hydrostatic model, of a radiation condition at the shoreline (equivalent to that which might be employed at $x = -\infty$ in the case of the constant-depth diffraction problem). Indeed, as pointed out, if the assumption of pure incoming wave at $+\infty$ were retained, it would be tantamount physically to imposing a right-propagating wave (emanating

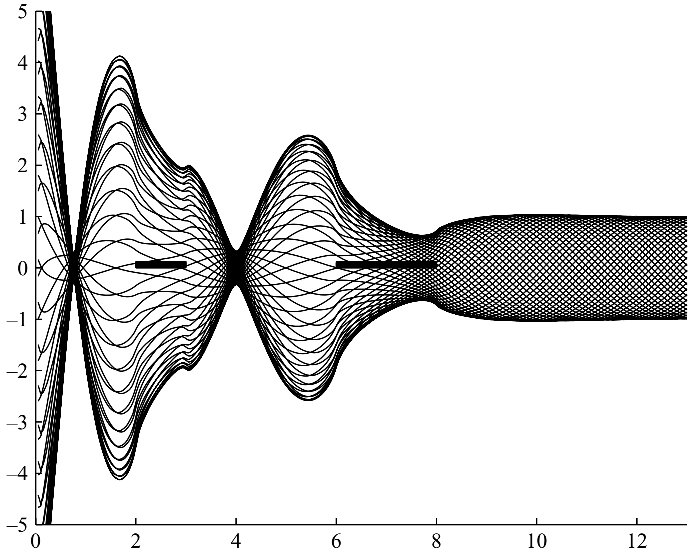

FIGURE 11. Wave envelope for the case of pure incoming progressive wave on two plates from ∞ with the narrow plate nearer the shore; beach angle $=\pi/8$.

from the log singularity at the origin) which is such that it would precisely cancel the true reflected wave by the plate.

To rectify the situation in the present problem, we therefore propose the construction of a suitable radiation condition at the shore to prevent this fictitious wave from emerging. Full details are deferred to Appendix E, but the result is that if we define a radiation operator $\boldsymbol{L}_R$ by

$$\boldsymbol{L}_R \equiv (\pi - \mathrm{i}\lambda_0 + \mathrm{i}\log R)\, R\frac{\partial}{\partial R} - \mathrm{i},$$

(where λ_0 is a constant defined in § E.2 of Appendix E) it follows, for a left-travelling wave Φ_I, that $\lim_{R\to 0}\{\boldsymbol{L}_R[\Phi_I]=0\}$ and that therefore the appropriate radiation condition which ensures no waves emanating from $R=0$ is

$$\lim_{R\to 0} \boldsymbol{L}_R[\Phi] = O(R(\log R)^2), \tag{7.1}$$

where

$$\Phi = Re\{\phi(R,\theta)\mathrm{e}^{\mathrm{i}t}\}.$$

At $R=+\infty$ we can take the more explicit radiation form

$$\phi \sim \left[(1+Q)\varphi_\infty^{(r)} + \mathrm{i}(1-Q)\varphi_\infty^{(s)}\right], \tag{7.2}$$

which describes a unit-amplitude incoming progressing wave and a reflected wave of amplitude $|Q|$. Then $A=(1+Q)$, $B=\mathrm{i}(1-Q)$ and the equivalent of (5.5) is

$$\mathrm{i}(1-Q) - c\pi/\sqrt{k} = 4\pi\kappa \int_a^{a+\ell} \phi(R,0)\varphi^{(r)}(R)\,\mathrm{d}R.$$

At this point it is convenient to introduce the notations

$$\langle p,q\rangle = \int_a^{a+\ell} \phi^{(p)}(R,0)\varphi^{(q)}(R)\,\mathrm{d}R$$

and

$$z_{qq} = 4\pi\kappa \mathrm{i}\langle q, q\rangle; \; z_{pq} = 4\pi\kappa\langle p, q\rangle, \, p \neq q.$$

With the help of this, the above condition is just

$$\mathrm{i}(1 - Q) - 4\pi\kappa\langle r, r\rangle(1 + Q) = T(1 + 4\pi\kappa\langle s, r\rangle), \tag{7.3}$$

whilst the radiation condition at the origin is written

$$(1 + Q)\boldsymbol{L}_R[\varphi^{(r)}] + T\boldsymbol{L}_R[\varphi^{(s)}] - 4\pi\kappa \mathrm{i}\sqrt{k}((1 + Q)\langle r, s\rangle + T\langle s, s\rangle) = 0. \tag{7.4}$$

(Note that this result required the near-field z-asymptotics of $G(R, 0|z, 0)$ as discussed earlier.)

Meanwhile, in the general case, the asymptotics which yielded (7.1) also yield $L_R[\varphi^{(r)}] = -\mathrm{i}\sqrt{k}, \; L_R[\varphi^{(s)}] = \sqrt{k}$; so T is now readily eliminated to yield a formula for the reflection coefficient. After simplification, noting also that $z_{rs} = z_{sr}$ (This follows readily from the properties of the operator K and changing orders of integration (see e.g. Porter & Stirling 1996, p. 104).) this is

$$Q = \frac{(1 - z_{ss})(1 + z_{rr}) - (1 + z_{sr})^2}{(1 - z_{ss})(1 - z_{rr}) + (1 + z_{sr})^2}. \tag{7.5}$$

This particular form, of course, assumes that, in the absence of the plate, there would be no propagated reflection. That can be altered by adding on, to the radiation conditions, the near- and far-field asymptotics of any given right-travelling wave solution to the fundamental problem. In the present work though we focus on the effect on $|Q|$ of varying the plate length and so will retain (7.5). A computation for two different situations is given here. Firstly in figures 12 and 13 are shown the wave envelopes for a given plate on a steep beach and a shallow beach respectively (here shown also in comparison with the case in which the plate is removed). Note how the beach itself produces a trapped reflected wave. Secondly, by way of verification of the reflection formula, the graph of $|Q|$ is shown against varying plate lengths for three differently sloping beaches in figure 14. In order to effect something of a comparison with another model, the shoreward tip of the plate has in each case been anchored at the depth $Kh = 0.5$; $K = \omega^2/g$. The finite dock on uniform depth was investigated in Linton (2001) for that value of Kh, and the solid line is computed from (4.4) therein.

The whole approach is further validated by an examination of the energy flux (see § E.5 of Appendix E) across boundaries in both the near and far fields. This results in the useful identification of T above as a formal 'transmission coefficient', and indeed, solving for T in the above, there readily follows $|Q|^2 + |T|^2 = 1$.

The numerical evidence at hand suggests that eigensolutions are possible. For the case $\lambda = 0$, the value of $\det(I - K^*)$, where K^* is the numerical discretization matrix of K, has been computed over the parameter domain $[\ell \times a] = [0.5 : 1.5 \times 0.1 : 10]$, and some results are depicted in figures 15 and 16. Note in particular that the ultimate spacings between successive curves of zero determinant, as functions of plate length, appear to approach π, the semi-wavelength of undisturbed wave motion at infinity. Clearly, further investigation of this would be of interest. In particular, an explanation is required for the (counter-intuitive) inclination of the contours, given that fundamental wavelengths are shorter in shallower water.

8. Summary

The author's previous work (Ehrenmark 2003) has provided the grounding for the construction of a numerically feasible Green's function to deal with a class of

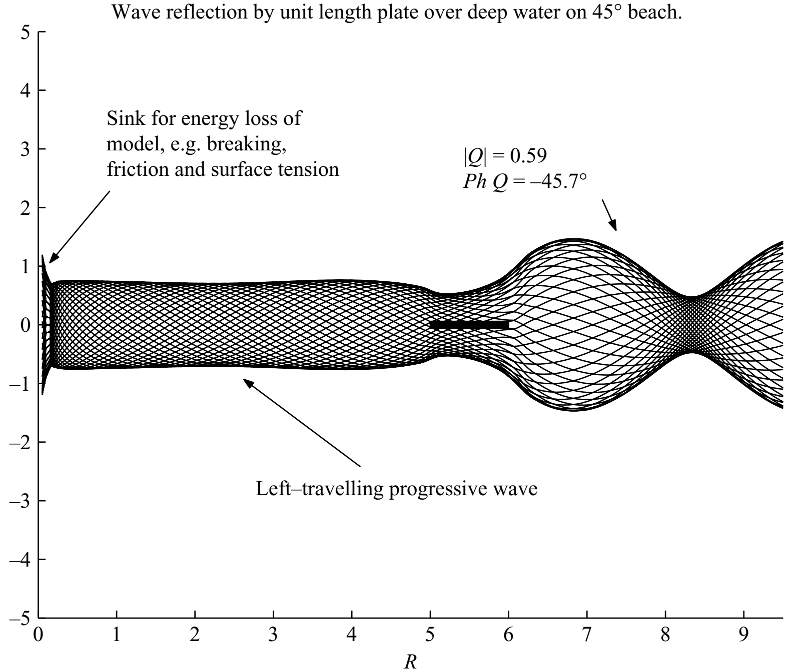

FIGURE 12. Wave envelope on a steep beach, developed by implementation of the radiation conditions; beach angle $= \pi/4$.

diffraction problems of small-amplitude two-dimensional waves on a plane beach. In this particular work, devoted in part to verification of the approach, attention has focused on the effect, on infinitesimal wave motion, of the placement of a finite rigid plate (dock) of arbitrary length at an arbitrary position on the water surface. The method reduces to the solution of a Fredholm integral equation of the second kind with a real symmetric kernel. The results are achieved by application of the trapezoidal rule suitably modified to engage the weakly singular elements at essentially the same level of accuracy as the remaining elements, although it is recognized that more accurate (but also more labour-intensive) schemes can be constructed. Results are obtained for both the case in which the plate is anchored at the shoreline (origin) and the case in which it is anchored some distance out at sea. The case of two disjointed plates is also considered. These indicate that, in the former case, conditions underneath are relatively quiescent but that if even a comparatively small amount of clear water is allowed, there can be significant oscillation amplitude near the shoreline or, indeed, in the small gap between two plates. Cases of peaks appear to occur as the anchor point is progressively pushed seawards. The occurrence of these appears to be largely in tune with the fundamental wavenumber for unrestricted motion on the beach.

A significant outcome of the present work is the construction and implementation of a radiation condition at the origin to facilitate treatment of the diffraction problem on a beach in a conventional way. This has allowed comparison with an existing model for uniform depth, and the results appear to be excellent for shallow beaches. For steeper beaches the indication is that reflection is gradually reduced.

It should be possible to extend the application to having any finite number of separate plates on the surface, thus allowing investigation of linear resonances between the spacings and the fundamental wavenumbers. It should also be possible to extend

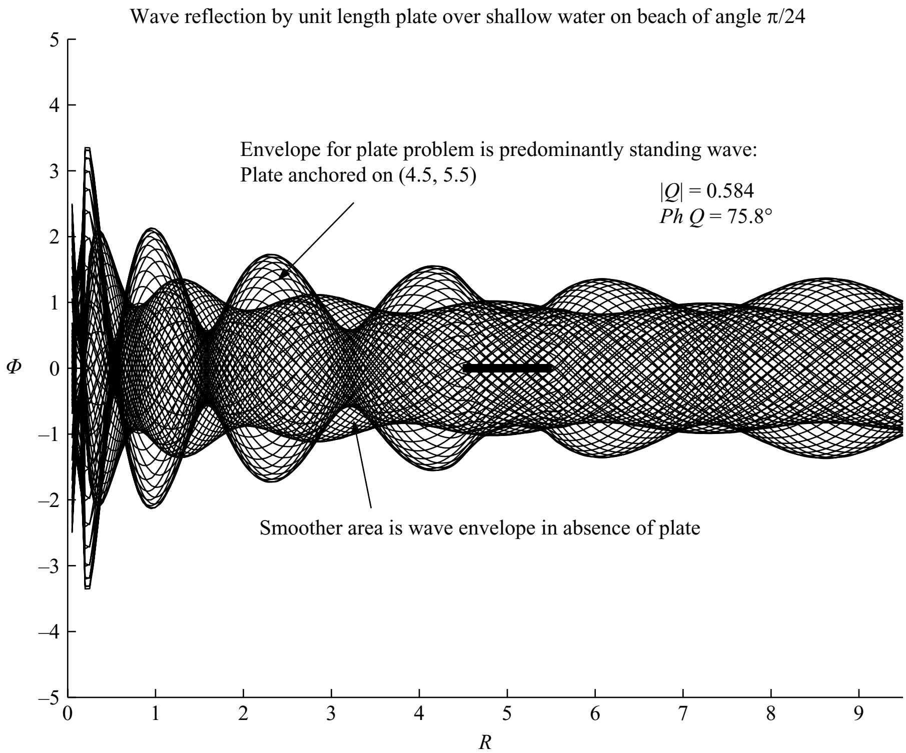

FIGURE 13. Wave envelope comparison on a shallow beach, developed by implementation of the radiation conditions. Plate resides on [4.5, 5.5]. Also shown is envelope for motion without the plate. Beach angle $=\pi/24$.

the model to calculate the heave-and-pitch motion of the plate if it is allowed to float freely on the surface. This extension was considered by Hermans (2003) in the case of motion over a flat bottom.

Hermans (2003) also extended his model to the case of a flexible platform (see also Chung & Linton 2003, 2005 for other models), and it is suggested that a similar extension to the present work might well be possible with a consideration also of the effects on flexural ice sheets near a shoreline being subjected to the effects of oncoming waves. This problem is well known to be of fourth order in the boundary conditions (see e.g. Porter & Porter 2004), but the present Green's function should be adapted to suit such a problem. Other possible applications include the effect of small-amplitude disturbances on the bed, distortions of the bed and even the problem over a Booij-like ramp (see Booij 1983), where a slope communicates with a flat-bottom section. In the case in which the slope extends indefinitely, it should be comparatively straightforward to develop a solution based on the present model.

Thanks are due to the host, the University of Reading, for providing facilities and in particular to Professor D. Porter for constant encouragement and a number of extremely valuable discussions and written contributions including comments on an earlier draft. The referees are also thanked for their many useful suggestions.

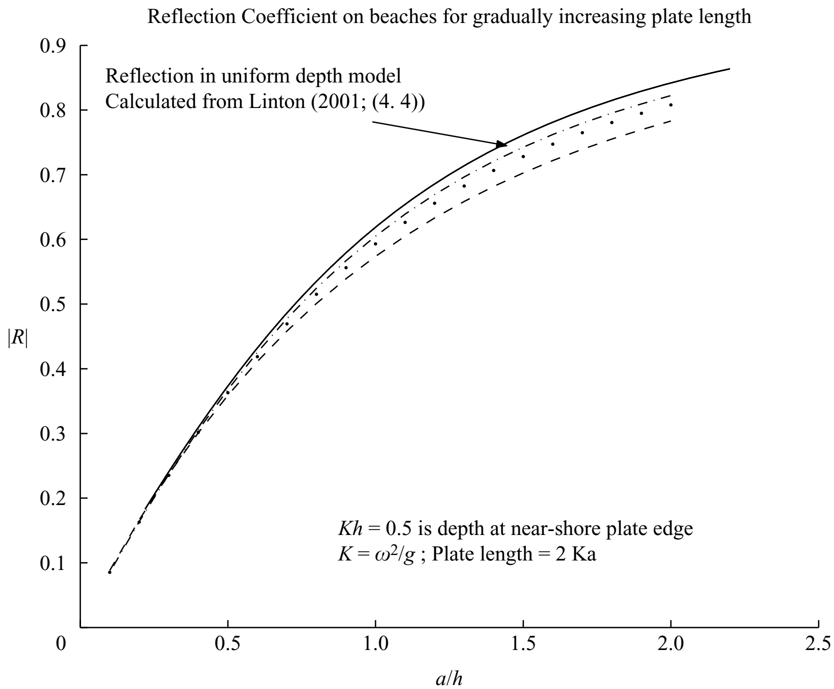

FIGURE 14. Reflection curves for three beaches with the implementation of the shoreline radiation condition. Broken curve: $\alpha = \pi/4$; dotted curve: $\alpha = \pi/12$; dash-dotted curve: $\alpha = \pi/24$. Solid curve computed from (4.4) in Linton (2001) with normal incidence and water depth at $Kh = 0.5$. In each case the shoreward tip of the plates was placed at that same depth.

Appendix A. The solution of the difference equation

The original difference equation on which the Green's function is based, is

$$d(s) - d(s-1) = -\frac{2\pi\rho^s \cos s(\gamma + \alpha)}{\sin \alpha s \,\Gamma(s+1) B_k(s) \sin \pi s},$$

where B_k has a zero at $s = -k$ and simple poles at $s = 0, k+1$, but is otherwise free of zeroes and poles in $-k < Re\ s < k+1$. Also $B_k(s)B_k(1-s) = \pi/\sin \pi s$ (see Ehrenmark 2003).

One of many procedures to solve the equation is to assume first that $d(s)$ is analytic in $\sigma - 1 < Re\ s < \sigma$, where, in this case, we take σ in the open domain (0, 1). Let L be the boundary of the rectangle with vertices $(\sigma, \pm iY)$, $(\sigma - 1, \pm iY)$. Then, describing L anticlockwise, a particular integral is

$$d_1(s) = -\frac{1}{2i} \int_L d(\tau) \cot \pi(s - \tau)\, d\tau. \tag{A1}$$

Then provided that the net contribution on the two line segments $\mathrm{Im}\tau = \pm Y$ is null as $Y \to \infty$, we find that

$$d_1(s) = -\pi i \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{\cos \tau(\gamma + \alpha)\rho^\tau \cot \pi(s - \tau)}{\sin \tau\alpha \ \Gamma(\tau + 1) B_k(\tau) \sin \tau\pi}\, d\tau.$$

This particular integral is readily seen to define an analytic function in $-1 < Re\ s < 1$. To observe the nullity *a posteriori* one observes readily that $d_1(s) = \mp\pi I + O(1)$ on

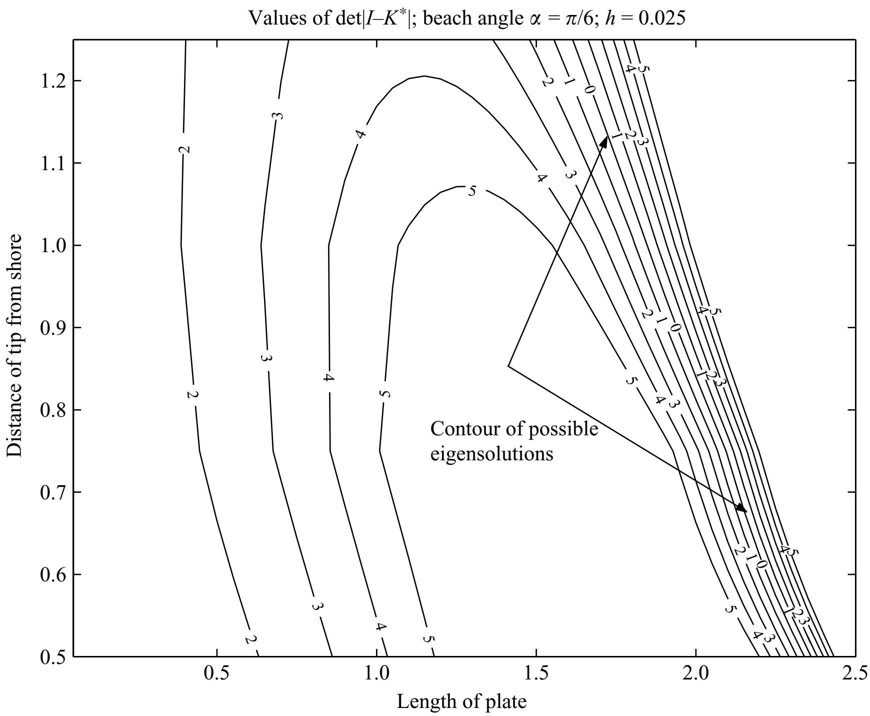

FIGURE 15. Experimental evidence for the possibility of eigensolutions shown by computation of the determinant of $I - K^*$; short plates.

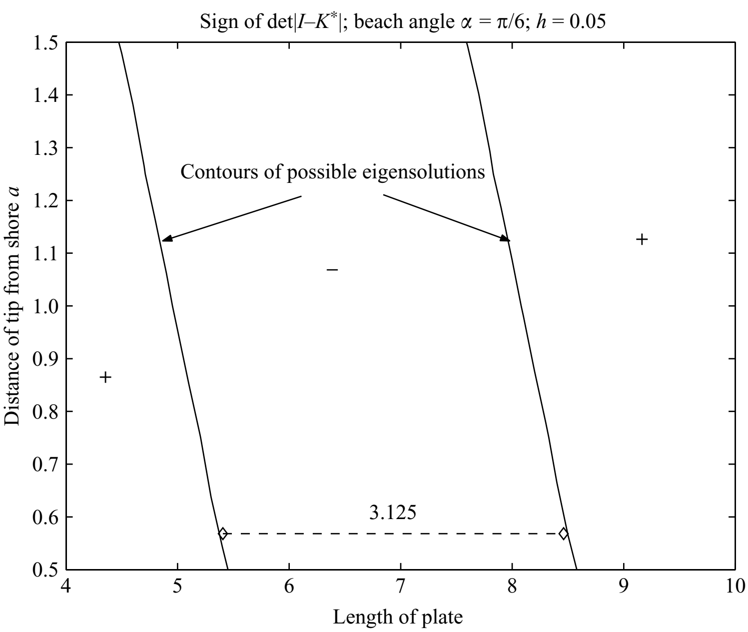

FIGURE 16. Experimental evidence for the possibility of eigensolutions shown by computation of the determinant of $I - K^*$; longer plates.

$\mathrm{Im} s = \pm Y$, where

$$I = \int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} \frac{\cos\tau(\gamma+\alpha)\rho^{\tau}}{\sin\tau\alpha\ \Gamma(\tau+1)B_k(\tau)\sin\tau\pi}\,\mathrm{d}\tau,$$

an integral which converges absolutely (for all ρ, γ in our domain) if $\sigma > 1/2$. It is then an easy matter to see that the O(1) contributions to the integrals in (A1) on $\mathrm{Im}\tau = \pm Y$ cancel out.

To achieve a Green's function which is bounded at $R=0$, we need to remove the logarithmic singularity there arising from $G^{(0)}$. Thus the requirement of d is a simple pole at $s=0$ with appropriate residue. Since d_1 above is regular at $s=0$ we need to add a 'complementary solution', which in this case needs to be a periodic function of real period unity, that carries this pole structure. It follows (extending partly the notation as used in the main text) that the appropriate solution is

$$d(s|\rho) = -\frac{2\pi}{\sqrt{\alpha}}\cot\pi s + d_1(s|\rho).$$

Finally, in order to assess the behaviour of d as $\rho \to \infty$, it is necessary to move the contour for d_1 to the left of the imaginary axis and take account of the residue at $s=0$. This residue exactly cancels the 'complementary function' added. Thus the result

$$d(s|\infty) \sim O(1)$$

is valid only if $Re\ s \leqslant -\delta < 0$ for some small positive δ, since the point $\tau = s$ is always to the left of the line of integration.

Appendix B. Symmetry of Green's function

The Green's function is defined by (4.1):

$$G(R,\theta|\rho,\gamma) = \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} R^{-s}B_k(s)\sin\pi s\ \Gamma(s)d(s|\rho,\gamma)\frac{\cos s(\theta+\alpha)}{\cos s\alpha}\,\mathrm{d}s + G^0(\zeta|z),$$

where, with $\sigma - 1 < Re\ s < \sigma$, $d(s)$ may be given by

$$d(s|\rho,\gamma) = -\frac{2\pi\rho^s\cos s(\gamma+\alpha)}{\Gamma(s+1)B_k(s)\sin s\alpha\sin\pi s} - \pi\mathrm{i}\int_{\sigma-1-\mathrm{i}\infty}^{\sigma-1+\mathrm{i}\infty}\frac{\cos\tau(\gamma+\alpha)\rho^{\tau}\cot\pi(s-\tau)}{\sin\tau\alpha\ \Gamma(\tau+1)B_k(\tau)\sin\tau\pi}\,\mathrm{d}\tau, \tag{B1}$$

which follows from (4.3) by passing the contour across the two poles at $\tau=0$ and $\tau=s$. With the transformation $\tau = -t$ in the second integral and with $\sigma = 1/2$ an alternative expression is obtained:

$$d(s|\rho,\gamma) = -\frac{2\pi\rho^s\cos s(\gamma+\alpha)}{\Gamma(s+1)B_k(s)\sin s\alpha\sin\pi s} - \frac{\mathrm{i}}{\pi}\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty}\frac{\cos\tau(\gamma+\alpha)\rho^{-\tau}\Gamma(\tau)B_k(\tau)\cot\pi(s+\tau)\sin\pi\tau}{\cos\tau\alpha}\,\mathrm{d}\tau.$$

In the above, use has been made of the continuation formula $B(s+1) = B(s)\tan s\alpha$. When these two terms are substituted for d in the definition of G, this can be written

$$G(\zeta|z) = \mathrm{i}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\left(\frac{R}{\rho}\right)^{-s}\frac{\cos s(\gamma+\alpha)\cos s(\theta+\alpha)}{s\sin s\alpha\cos s\alpha}\,\mathrm{d}s + F(\zeta|z) + G^0(\zeta|z), \tag{B2}$$

where F is the repeated integral

$$F(\zeta|z) = -\frac{1}{2\pi^2}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{d}s \int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} R^{-s}\rho^{-\tau} g(s,\tau)\,\mathrm{d}\tau$$

and g is the symmetric function

$$g(s,\tau) = \frac{B_k(s)B_k(\tau)\Gamma(s)\Gamma(\tau)\sin\pi s\sin\pi\tau\cos\tau(\gamma+\alpha)\cos s(\theta+\alpha)\cot\pi(s+\tau)}{\cos\tau\alpha\cos s\alpha}.$$

The symmetry of F follows from the observation that we can take $c = 1/2 = \sigma$ in the outer integral. It is also necessary to have the asymptotics of B_k and Γ to confirm, by dominated convergence, that the interchange of the infinite integrals is justified. There is exponential dominance, in both variables, at every interior point of the domain D (see e.g. Ehrenmark 1987 for details).

That leaves us (from (B2)) to establish the symmetry of G_1 defined by

$$G_1(\zeta|z) = \mathrm{i}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\left(\frac{R}{\rho}\right)^{-s}\frac{\cos s(\gamma+\alpha)\cos s(\theta+\alpha)}{s\sin s\alpha\cos s\alpha}\,\mathrm{d}s + G^0(\zeta|z).$$

Pass this integral over the double pole at $s = 0$, and then make the transformation $s = -S$. The result is

$$G_1(\zeta|z) = \frac{2\pi}{\alpha}\log(R/\rho) + \mathrm{i}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\left(\frac{R}{\rho}\right)^{s}\frac{\cos s(\gamma+\alpha)\cos s(\theta+\alpha)}{s\sin s\alpha\cos s\alpha}\,\mathrm{d}s + G^0(\zeta|z).$$

Now map $(\zeta|z) \longrightarrow (z|\zeta)$ in this last expression and subtract from the previous to form

$$G_1(\zeta|z) - G_1(z|\zeta) = G^0(\zeta|z) - G^0(z|\zeta) + \frac{2\pi}{\alpha}\log(R/\rho) = 0.$$

Thus G_1 is symmetric, and hence so is G. The symmetric form of G_1 is readily seen to be

$$G_1(\zeta|z) = \log\left|\frac{\left(\zeta^{\frac{\pi}{2\alpha}} - z^{\frac{\pi}{2\alpha}}\right)\left(\zeta^{\frac{\pi}{2\alpha}} - \bar{z}^{\frac{\pi}{2\alpha}}\right)}{(R\rho)^{\frac{\pi}{\alpha}}}\right| + \mathrm{i}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\left\{\left(\frac{R}{\rho}\right)^{s} + \left(\frac{R}{\rho}\right)^{-s}\right\} \times \frac{\cos s(\gamma+\alpha)\cos s(\theta+\alpha)}{s\sin 2\alpha s}\,\mathrm{d}s.$$

Appendix C. Dock problem computation

Setting $k = 1/2$ in (2.5) allows part of the integrand to be broken by partial fractions. This facilitates, through the use of Kummer's result (Whittaker & Watson 1952, p. 250, Example 3), the simpler expression

$$B_{\frac{1}{2}}(s) = \left(\frac{\pi}{\sin\pi s}\right)^{\frac{1}{2}}\exp -\frac{1}{4}\int_0^{\infty}\frac{\sinh(s-\frac{1}{2})t}{\cosh^2\frac{t}{4}}\frac{\mathrm{d}t}{t}.$$

An alternative description is readily seen to be

$$B_{\frac{1}{2}}(s) = \left(\frac{\pi}{\sin\pi s}\right)^{\frac{1}{2}}\exp -2\pi\int_0^{s-\frac{1}{2}}\frac{\tau}{\sin 2\pi\tau}\,\mathrm{d}\tau.$$

To compute the value of $B'_{1/2}(1)$ we need to use the original expression (2.5) again, as Kummer's result is invalid at $s = 1$. Then, by direct differentiation of the formula, we find

$$\frac{B'_{\frac{1}{2}}(1)}{B_{\frac{1}{2}}(1)} = \Psi(1) - I + \int_0^\infty \frac{\mathrm{d}t}{\mathrm{e}^t - 1}\left\{\frac{\mathrm{e}^t + 1}{\mathrm{e}^{t/2} + 1} - \mathrm{e}^{-t}\right\}, \tag{C1}$$

where Ψ is the digamma function and

$$I = \int_0^\infty \mathrm{e}^{-t}\left\{\frac{1}{t} - \frac{1}{\mathrm{e}^t - 1}\right\}\mathrm{d}t = \Psi(2).$$

The integral in (C1) is readily evaluated by substitution and partial fractions resulting in

$$\frac{B'_{\frac{1}{2}}(1)}{B_{\frac{1}{2}}(1)} = 2 + \Psi(1) - \Psi(2) = 1.$$

It is also required to have easy numerical access to values of $B_{1/2}((3/4) + \mathrm{i}y)$ for all real y. Following from the second expression for $B_{1/2}(s)$ above, we have readily $B_{1/2}((3/4) + \mathrm{i}y) = (\pi/\sin\pi s)^{1/2}\exp I_2$, where

$$I_2 = -\frac{\boldsymbol{G}}{\pi} - \frac{\mathrm{i}}{4}\tan^{-1}\sinh 2\pi y + 2\pi\int_0^y \frac{x}{\cosh 2\pi x}\,\mathrm{d}x,$$

where $\boldsymbol{G}$ is Catalan's constant 0.915965594.... The integral term can also be replaced by a uniformly valid expansion so that, on $s = (3/4) + \mathrm{i}y$,

$$B_{\frac{1}{2}}(s) = \left(\frac{\pi}{\sin\pi s}\right)^{\frac{1}{2}}\exp\left[-\frac{\mathrm{i}}{4}\tan^{-1}\sinh 2\pi y - \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-)^k}{1+2k} \times\left\{2\pi|y| + \frac{1}{1+2k}\right\}\mathrm{e}^{-2\pi|y|(1+2k)}\right]. \tag{C2}$$

From this values on $\tau = (1/4) + \mathrm{i}x$ are readily found using the folding formula $B_k(s)B_k(1-s) = \pi/\sin\pi s$. Thus

$$B_{\frac{1}{2}}\left(\frac{1}{4} + \mathrm{i}x\right) = \left(\frac{\pi}{\sin\pi\left(\frac{1}{4} + \mathrm{i}x\right)}\right)^{\frac{1}{2}}\exp\left[-\frac{\mathrm{i}}{4}\tan^{-1}\sinh 2\pi x + \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-)^k}{1+2k} \times\left\{2\pi|x| + \frac{1}{1+2k}\right\}\mathrm{e}^{-2\pi|x|(1+2k)}\right].$$

For very small $|y|$, say $|y| < 1/2\pi$, it is probably most practical to use the Taylor expansion of I_2. This requires (Art. 2.477.16, item no. 16, p. 126 in Gradshteyn & Ryzhik 1965)

$$\int_0^y \frac{x}{\cosh x}\,\mathrm{d}x = \sum_{k=0}^{\infty}\frac{E_{2k}y^{2k+2}}{(2k+2)(2k)!},\ E_0 = 1,\ E_2 = -1,\ E_4 = 5,\ E_6 = -61, \ldots.$$

Appendix D. Numerical quadrature

The numerical quadrature routine used in this work is based on the trapezoidal rule and has evolved from the simple rule (a) in which the log term is integrated exactly only on the elements adjoining the weak singularity through a more general rule (b) of assuming that ψ takes constant values in each sub-interval and integrating

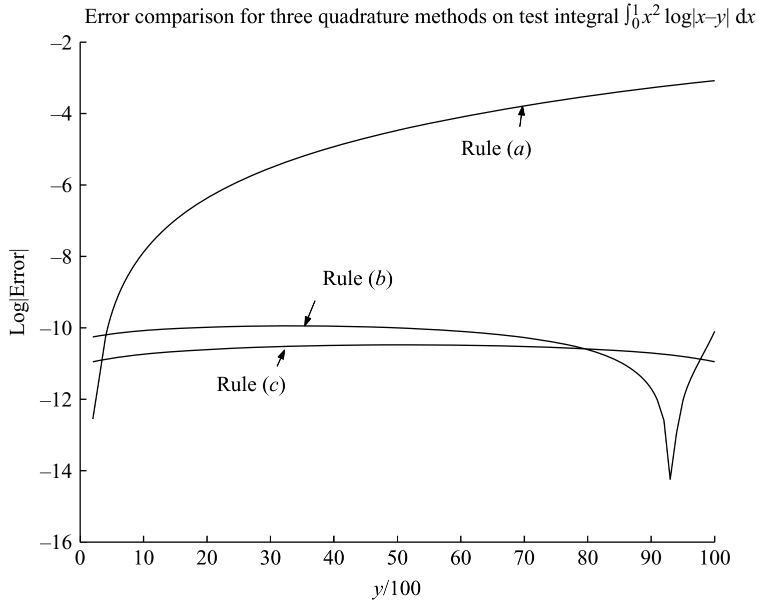

FIGURE 17. The absolute errors in formula (D1), using piecewise linear fit compared to midpoint rule function values.

every log term exactly, finally to a rule (c) assuming a piecewise linear approximation in each such interval. The fundamental formula for this, using the normal notation, is

$$I(x_j) = \int_a^b \psi(x) \log|x - x_j|\,\mathrm{d}x = \sum_{i=1}^{N} \int_{x_i}^{x_{i+1}} \psi(x) \log|x - x_j|\,\mathrm{d}x,$$

and so writing

$$f_{i,j} = \int_0^1 \log|x_i - x_j + \tau h|\,\mathrm{d}\tau, \quad g_{i,j} = \int_0^1 \tau \log|x_i - x_j + \tau h|\,\mathrm{d}\tau$$

we have, for this optimum rule (c)

$$I(x_j) \approx h \sum_{i=1}^{N} (f-g)_{i,j}\psi_i + g_{i,j}\psi_{i+1} = h \sum_{i=1}^{N+1} w_{i,j}\psi_i, \tag{D1}$$

where

$$w_{1,j} = f_{1,j} - g_{1,j}, \;\; w_{N+1,j} = g_{N,j} \text{ and } w_{i,j} = (f-g)_{i,j} + g_{i-1,j}.$$

Note that the weights only depend on $i - j$. With $k = i - j$ and $\sigma = sgn(k)$ we can write

$$f_{i,j} \equiv f(k) = -1 - k\log h|k| + (1+k)\log(h\sigma(1+k)), \;\; f(0) = -1 + \log h,$$
$$2g_{i,j} \equiv 2g(k) = -\frac{1}{2} + k - k^2\log(1 + 1/k) + \log(h\sigma(1+k)); g(-1) = -\frac{3}{4} + \frac{1}{2}\log h.$$

The results displayed in figure (17) show that rule (a) is substantially inferior and that rule (c) represents a further significant improvement with errors generally halved compared with rule (b). (The spike in (c) is due to the error changing sign and can therefore essentially be ignored.)

D.1. *Two plates*

It is possible to extend the method to multiple plates. Here we consider two plates of lengths $\ell_0 = M_0 h$ and $\ell_1 = M_1 h$, where the distances from shore to the near-shore tip of the plates are respectively $L_0 h$ and $(L_0 + M_0 + L_1)h$, with $(L_0, M_0, L_1, M_1) > 0$. If we then define the required computational grid on the surface by $x_i = (i-1)h,\ i = 1, 2, \ldots,\ N_1 + 1$, where $N_1 = L_0 + M_0 + L_1 + M_1 + L_2$, it follows that $L_2 h$ denotes the length of the clear water computational domain seaward of the second plate.

The integral equation will first need to be solved on the union of the two plates. Thus we introduce the $M_0 + M_1 + 2$ vector v_j by

$$\begin{aligned} v_j &= \phi(L_0 + j)h, \quad j = 1, 2, \ldots, M_0 + 1, \\ v_j &= \phi(L_0 + L_1 + j)h, \quad j = M_0 + 2, \ldots, M_0 + M_1 + 2. \end{aligned}$$

The discretization when the source points are on the first plate is therefore expressible in the form

$$\begin{aligned} v_j = f_{L_0+j} &+ \frac{\kappa h}{2\pi} \sum_{i=1}^{M_0+1} (\epsilon_i G_0(x_{L_0+i} | x_{L_0+j}) + w_{i,j}) v_i + \frac{\kappa h}{2\pi} \\ &\times \sum_{i=M_0+2}^{M_0+M_1+2} (\epsilon_i G_0(x_{L_0+L_1+i-1} | x_{L_0+j}) + w_{L_1+i-1,j}) v_i, \quad j = 1, 2, \ldots, M_0 + 1. \end{aligned}$$

Here f_j denotes the inhomogeneous term of the pertinent integral equation and $\epsilon_i = 1/2$ at each end point of summation but is otherwise unity. A similar form when the source points are on the second plate is

$$\begin{aligned} v_j = f_{L_0+L_1+M_0+j} &+ \frac{\kappa h}{2\pi} \sum_{i=1}^{M_0+1} (\epsilon_i G_0(x_{L_0+i} | x_{L_0+L_1+j}) + w_{i,L_1+j-1}) v_i + \frac{\kappa h}{2\pi} \\ &\times \sum_{i=M_0+2}^{M_0+M_1+2} (\epsilon_i G_0(x_{L_0+L_1+i-1} | x_{L_0+L_1+j-1}) + w_{i,j}) v_i, \ \ j = M_0 + 2, \ldots, M_0 + M_1 + 2. \end{aligned}$$

Appendix E. Shoreline radiation condition

Here the question is considered of how to derive and apply a radiation condition at the shoreline when an obstacle is placed to disrupt incoming waves.

E.1. *The constant-depth analogy*

The mapping in (Roseau 1976, pp. 312–328) for the case of a beach suggests an analogy between the beach problem and the more conventional problem of waves in a channel of uniform depth. In this analogy, the shoreline $R = 0$ would be mapped to $\xi = -\infty$, whilst the SWL at $R = \infty$ would be mapped similarly to $\xi = +\infty$.

In the channel problem, a one-parameter family of waves might be given by $\zeta_I = ARe\{\exp \mathrm{i}(kx + t)\}$, where A is arbitrary and k depends on the depth h. A second independent family is given by $\zeta_R = BRe\{\exp \mathrm{i}(-kx + t)\}$, where B is arbitrary.

The analogy for the beach case, using notations well-defined elsewhere, is that one family is given by

$$\zeta_I = ARe\{(\varphi^{(r)} + \mathrm{i}\varphi^{(s)}) \exp \mathrm{i}t\}$$

and the other by

$$\zeta_R = BRe\{(\varphi^{(r)} - \mathrm{i}\varphi^{(s)}) \exp \mathrm{i}t\}.$$

At this point both wave families are equally represented, and there is no curiosity about the fact that, for the beach problem, both ζ_I and ζ_R are acceptable without further restrictions being placed. In particular, one cannot expect the reflective property of the beach itself to be deterministic in this model. Moreover, it is easily argued from an observational standpoint that, without obstacles present, the degree of reflection observed at long distance is negligible for most beaches so that an appropriate condition might be $B = 0$ or at the most $B \ll A$. With an obstacle present, facing an incoming wave from $+\infty$ of unit amplitude, the convention is to solve the problem by the assertion that the diffraction can cause a wave travelling to the right only to the right of the obstacle, so that as $x \to -\infty$ the appropriate condition is that there must be no waves of the second family. A convenient way of stating this mathematically is in the form of the radiation condition

$$\lim_{x \to -\infty} \frac{\partial \phi}{\partial x} - \mathrm{i}k\phi = 0.$$

We now consider how this might be similarly done at the shoreline for the beach case.

E.2. *The beach asymptotics*

Remembering that the constant-depth radiation condition arises essentially from the limiting form of an eigenvalue problem where, as $\lim_{x \to -\infty}$, the only surviving eigenfunctions are of the type $\exp \pm \mathrm{i}kx$; all other roots having represented waves trapped by the obstacle, we now consider if a further analogy is appropriate for the beach. It would thus be prudent to examine the similar survival of terms in the expansions of $(\varphi^{(r)}, \varphi^{(s)})$ as $R \to 0$. The beach problem has not lent itself to a modal expansion; instead we expand the known classical solutions directly from their integral expansions as in Ehrenmark (1996). Therein we find

$$\varphi^{(s)} \sim \sqrt{\left(\frac{1}{2\pi\alpha}\right)} (\log R - \lambda_0 + d_1 R \log R + O(R)) \tag{E1}$$

and

$$\varphi^{(r)} \sim \sqrt{\left(\frac{\pi}{2\alpha}\right)} (1 + d_1 R + O(R^2)), \tag{E2}$$

where d_1, λ_0 are independent of R. Specifically, $d_1 = -\cos(\theta + \alpha)/\sin\alpha$ and $\lambda_0 = \Psi(1) - \alpha \sum_{j=1}^{M-1} \tan j\alpha$, where $\alpha = \pi/2M$; $M \in \mathbf{N}$ is the beach angle; and Ψ is the digamma function.

E.3. *Directed waves*

The near-field asymptotics of a directed progressing wave ϕ will therefore be of the form

$$\phi \sim B\mathrm{e}^{\mathrm{i}t} \left(1 + \frac{\mathrm{i}\mu\lambda_0}{\pi} - \frac{\mathrm{i}\mu(1 + d_1 R)}{\pi} \log R\right) + O(R), \tag{E3}$$

where $\mu = 1$ for a right-travelling wave and $\mu = -1$ for a left-travelling wave.

In the channel case, the radiation condition is essentially a differential operator which is null for every left-travelling wave but not for any right-travelling wave. We seek a similar scenario in the beach case.

E.4. *A radiation condition*

It is readily seen from (E3) that

$$\pi R \frac{\partial \phi}{\partial R} \sim -\mathrm{i}\mu B \mathrm{e}^{\mathrm{i}t} + O(R \log R),$$

and so by substitution therein, also

$$\frac{-\pi}{\mathrm{i}\mu} R \frac{\partial \phi}{\partial R} \left(1 + \frac{\mathrm{i}\mu\lambda_0}{\pi} - \frac{\mathrm{i}\mu(1 + d_1 R)}{\pi} \log R\right) - \phi = O(R \log R). \tag{E4}$$

Simplifying and setting $\mu = -1$ we obtain the radiation condition which will filter out waves emerging from the origin (i.e. travelling from left to right) in the form

$$\lim_{R\to 0} \left\{ R \frac{\partial \phi}{\partial R} (\pi - \mathrm{i}\lambda_0 + \mathrm{i} \log R) - \mathrm{i}\phi \right\} = O(R(\log R)^2) \tag{E5}$$

If the operator on the left-hand side above is applied to the right-travelling wave, the result is $-2\mathrm{i}\sqrt{k}$. Thus, if we define a shoreline radiation operator by

$$\boldsymbol{L}_R \equiv R(\pi - \mathrm{i}\lambda_0 + \mathrm{i} \log R) \frac{\partial}{\partial R} - \mathrm{i},$$

then we have

$$\lim_{R\to 0} \{\boldsymbol{L}_R[\zeta_I] = 0,\ \mathrm{L}_R[\zeta_R] = -2\mathrm{i}B\sqrt{k}\}.$$

E.5. *Energy flow and the transmission coefficient*

In connection with the above, it may be observed that the period mean energy flux across a vertical plane at great distance is given by $(1/4)(1-|Q|^2)$ (for a full discussion see Art. 237, pp. 382–383 in Lamb 1993) using the formulation in (7.2). If a similar computation is made on a small arc of radius ϵ around $R = 0$, it is found that the mean flux across this curve is

$$\frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}t\ \epsilon \int_{-\alpha}^{0} \left[\frac{\partial \Phi}{\partial t} \frac{\partial \Phi}{\partial R}\right]_{R=-\epsilon} \mathrm{d}\theta.$$

After averaging, noting the ansatz used for Φ and the near-field asymptotics of $\phi^{(r)}, \phi^{(s)}$, this simplifies to

$$\frac{1}{4}(1 + 2 < rs >)\mathrm{Im}(\overline{A}T).$$

It is further readily verified, using (7.3) and (7.4), that this reduces to $(1/4)|T|^2$. Thus by energy conservation, we have the desired result in the traditional form

$$|T|^2 + |Q|^2 = 1,$$

leading to the interpretation that T represents the transmission coefficient. Separate numerical confirmation of T and Q (over a wide range of plate lengths) shows agreement to three figures when the step length $h = 0.05$ is used in the integration routines.

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